N-63-2-5

ASD-TDR-62-973

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ADVANCED CONCEPTS OF STOCHASTIC PROCESSES AND STATISTICS FOR FLIGHT VEHICLE VIBRATION ESTIMATION AND MEASUREMENT

DECEMBER 1962

AERONAUTICAL SYSTEMS DIVISION

(PREPARED UNDER CONTRACT NO. AF33(657)-7459 BY THOMPSON RAMO WOOLDRIDGE INC., RW DIVISION, CANOGA PARK, CALIFORNIA

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NOTICES

ASD-TDR-62-973

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TECHNICAL DOCUMENTARY REPORT NO. ASD-TDR-62-973

December 1962

Flight Dynamics Laboratory
Aeronautical Systems Division
Air Force System Command
Wright-Patterson Air Force Base, Ohio

Project No. 1370, Task No. 137009

(Prepared under Contract No. AF33(657)-7459 by
Thompson Ramo Wooldridge Inc., RW Division,
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FOREWORD

The research work in this report was performed by Thompson Ramo Wooldridge Inc., RW Division, Canoga Park, California, for the Flight Dynamics Laboratory, Directorate of Aeromechanics, Deputy for Technology, Aeronautical Systems Division, Wright-Patterson Air Force Base, under AF Contract Nr. AF(657)-7459. This research is part of a continuing effort to obtain economical methods of vibration prediction, control and measurement for flight vehicles which is part of the Air Force Systems Command's Applied Research Program 750A, the Mechanics of Flight. The Project Nr. is 1370, "Dynamic Problems in Flight Vehicles," and Task Nr. is 137009, "Methods of Vibration Prediction, Control and Measurement." C. L. Pao of the Flight Dynamics Laboratory was the Project Engineer.

The authors express their appreciation for the contribution of C. L. Pao, R. N. Bingman, and O. R. Rogers. Contributing personnel of Thompson Ramo Wooldridge Inc., RW Division, include J. S. Bendat, L. D. Enochson, G. H. Klein, and A. G. Piersol.

ABSTRACT

This report presents results from theoretical and experimental studies on random signal estimation and measurement pertinent to flight vehicle vibration problems. The report is divided into two self-contained parts corresponding to the theoretical and experimental programs.

Part I, Theoretical Studies of Random Signal Estimation, develops new basic mathematical ideas for nonstationary data analysis. The discussion includes methods for estimating mean values and mean square values of nonstationary data, correlation and spectral properties of nonstationary data, and various input-output relations for passage of nonstationary data through linear systems. These results are illustrated on examples of physical interest. Other theoretical material discusses sampling formulas for gathering a minimum amount of sampled data for flight vehicle vibration problems. Analysis of variance procedures are explained for carrying out accurate statistical calculations of an over-all vibration environment at many points on an extended structure using sampled data obtained from one or many flights. A summary review is given of known technical results dealing with the response of nonlinear systems to random excitation.

Part II, Experimental Studies of Random Signal Measurements, investigates previously derived theoretical expressions for expected measurement uncertainties in zero crossing estimates, power spectra (mean square) estimates, and amplitude probability density estimates. These uncertainty expressions are tested experimentally by a properly designed statistical program which demonstrates how significant and accurate information can be obtained with a limited amount of data. New theoretical uncertainty formulas are derived which are subjected to experimental and statistical check. From these uncertainty formulas, the remaining experiments establish new statistical procedures for determining the fundamental properties of randomness, stationarity, and normality.

PUBLICATION REVIEW

This report has been reviewed and is approved.

FOR THE COMMANDER

Walter J. Mykytow

WALTER J. MYKYTOW
Chief, Dynamics Branch
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GLOSSARY OF SYMBOLS

$b(q)$	binomial probability density function
B	realizable bandwidth in cycles per second
c	physical definition: damping coefficient
c	number of columns in analysis of variance
C_i	total of ith column in analysis of variance
C. F.	"correction factor" due to mean in analysis of variance
e	length of unexpected event
E	expected value
f	physical definition: frequency in cycles per second
f	statistical definition: number of observations in a class interval
f_c	center frequency
f_0	frequency of a sine wave
F	variable with F distribution
F_{max}	ratio of largest to smallest variance in a set of several variances
F_i	expected number of observations in class interval i
g	acceleration due to gravity
g^2	mean square acceleration
$G(f)$, $G(\omega)$, $S(f)$, $S(\omega)$	power spectral density functions
$h(t)$	weighting function
$h(a, t)$	weighting function
H_0	null hypothesis
$H(f)$, $H(\omega)$, $\mathcal{H}(f, t)$	complex frequency response function
i, j	$\sqrt{-1}$
I	moment of inertia
j, i	$\sqrt{-1}$
k	physical definition: spring rate (spring constant)
k^*	upper percentage point of "studentized range"
k	statistical definition: number of groups of experiments or Tukey k-factor
K	physical definition: time constant of RC circuit, $K = RC$
K	statistical definition: tolerance factor

GLOSSARY OF SYMBOLS (Continued)

K_n	generalized stiffness
L	mean time between samples
ℓ	twice the radius of gyration of a beam's section
m	physical definition: mass
m	statistical definition: sample mean value
M_n	generalized mass
M	physical definition: concentrated mass
M	statistical definition: number of experiments
$\mathcal{M}$	expected number of maxima per unit time
ms_x, v	mean square value
n	number of degrees of freedom
N	sample size
p	percent value
$p_i(t)$	input signal-to-noise amplitude ratio at time t
$p_o(t)$	output signal-to-noise amplitude ratio at time t
$P_i(t)$	input signal-to-noise power ratio
$P_o(t)$	output signal-to-noise power ratio
$p(x)$	probability density function
P	physical definition: period, $(1/f)$
P	statistical definition: proportion, probability
$P(x)$	(cumulative) probability distribution function
q	probability of failure for binomial distribution
Q	mechanical Q, $(1/2\zeta)$
$Q_n(t)$	generalized input force
r	number of runs in "run test"
r	number of rows in analysis of variance
R_i	total of ith row in analysis of variance
$R(\tau)$, $R(\tau, T)$	stationary correlation functions
$R(t_1, t_2)$, $\mathcal{R}(t, \tau)$	nonstationary correlation functions
s^2	sample variance (s = sample standard deviation)
S	physical definition: stress level
SS	sum of squares
SS^*	unbiased variance estimate computed using SS
S. R.	sweep rate or scan rate
$S(f)$, $S(\omega)$, $G(f)$, $G(\omega)$	power spectral density functions

GLOSSARY OF SYMBOLS (Continued)

$S(f_1, f_2), \mathcal{S}(f, g)$	generalized (nonstationary) power spectral density
t	physical definition: time
t	statistical definition: variable with student's "t" distribution
T	physical definition: time interval
T	statistical definition: sample length (record length) or over-all total in analysis of variance
T_r	transmissibility
T_1	averaging time
v, ms_x	sample mean square value
$Var(x)$	variance of x (second moment about the mean)
W_{ij}	"within cell" totals in analysis of variance (the cell in the ith row, jth column)
x	any variable
$x(t), y(t), z(t)$	amplitude as a function of time
$\overline{x(t)}$	mean value of x(t)
$\overline{x}$	sample mean value
$\overline{x^2(t)}$	mean square value of x(t)
$\overline{x^2}$	sample mean square value
z	standardized normal variate (zero mean, unit variance)
$Z(\omega)$	mechanical impedance
α	physical definition: damping factor $\alpha = \frac{a}{2m}$
α	statistical definition: level of significance (i. e., probability of Type I error); or arbitrary level crossing of a random process
$(1 - \alpha)$	confidence coefficient
β	probability of Type II error
$(1 - \beta)$	power of test
$\gamma^2(f)$	coherence function at frequency f
γ_n	structural damping coefficient
δ	Dirac delta function
ϵ^2	normalized variance (normalized mean square error)
ϵ	normalized standard deviation (normalized standard error)
ϵ_s	strain

GLOSSARY OF SYMBOLS (Continued)

ζ	damping ratio (c/c_{cr})
Θ	phase angle
λ	number of standard deviations
$\lambda(f)$	spectral bandwidth
μ	population mean value
μ_v	population mean square value
ν_0	number of zero crossings
$\bar{\nu}_0$	expected number of zero crossings
ξ	generalized variable
ρ	covariance function
σ^2	population variance
σ	population standard deviation, $+\sqrt{\sigma^2}$
τ	time difference
χ^2	variable with chi-square distribution
ω	angular frequency, $2\pi f$
ω_n	natural angular frequency
$\wedge$	estimate of
$\langle \ \rangle$	ensemble average
$\sim$	distributed as
$\{ \ \}$	ensemble (collection) of functions
ψ	deflection

PART I

THEORETICAL STUDIES OF RANDOM SIGNAL ESTIMATION

1. INTRODUCTION AND SUMMARY OF MAIN RESULTS

1.1 REVIEW OF PREVIOUS CONTRACT

The present contract is a continuation of a previous contract sponsored by the Aeronautical Systems Division, Wright-Patterson Air Force Base, Ohio, on "The Application of Statistics to the Flight Vehicle Vibration Problem," ASD TR 61-123, December 1961, Contract AF33(616)-7434, (ASTIA No. AD 271 913). The general problem indicated by the title of the first contract is covered from several directions, and results are developed which are applicable to many other classes of physical data besides vibration data. A brief summary of the objectives and results of this previous work will be given here.

The objectives of the above contract were to determine the vibration environment at an arbitrary single point on a structure of a flight vehicle during its entire operational life history. No limitations were placed on the nature of the structure, whether linear or nonlinear, whether it had a complicated geometrical configuration, or whether there were complicated feedback loops and interconnections. Also, no limitation was placed on the nature of the various exciting forces that were acting upon the structure, either as to their location, or statistical nature. Some of these exciting forces might be transient, some might be localized, others distributed, some with stationary characteristics, others quite random or completely nonstationary. The main objective of the analysis is that on the basis of empirical information, a complete scientific statistical evaluation of the important characteristics contained in the data has to be made. It is required to predict the actual vibration environment that is encountered or that could occur under similar conditions so as to assist future fatigue and reliability investigations.

Engineering aspects of four different categories of aircraft and missiles were considered, namely (1) short-term missiles, (2) space vehicles and probes, (3) long-term missiles, and (4) rocket or jet powered aircraft. Each of these types were broken down into significant flight phases, and analyzed as to their dominant vibration and acoustic sources. Results were

Manuscript released by the authors 31 October 1962 for publication as an ASD Technical Documentary Report.

developed for the response of structures to periodic and random disturbances, and a survey was made of known experimental data obtained by various investigators.

Principal theoretical activites of the contract were divided into three phases as follows:

Phase I	Development of sampling techniques for decreasing the amount of data to be gathered for later detailed analysis.
Phase II	Development of statistical techniques for testing fundamental assumptions in data of (a) randomness, (b) stationarity, and (3) normality.
Phase III	Detailed examination of desired measurements for different applications such as amplitude probability density functions, correlation functions, power spectral density functions, etc.

During the course of the investigation, many mathematical and statistical ideas were explored in a broad way so as to be applicable to physical problems, such as occur in structural design problems. Various statistical concepts, tables, and curves were presented, including the normal (Gaussian) distribution, Chi-square distribution, "t" distribution, "F" distribution, quality control procedures, analysis of variance techniques, multiple regression techniques, etc. Detailed procedures were developed for carrying out: (1) a complete analysis on results that would be obtained from a single experiment, and (2) a complete analysis of appropriate results from a collection of experiments.

In addition to the theoretical work, the contract also involved carrying out an instrumentation study of equipment available today for making desired measurements, with emphasis on their practical physical limitations and accuracies. An experimental program was outlined for verifying the statistical procedures that were developed and the various instrumentation accuracies.

Many physical results on the response of structures under various conditions were also obtained during the course of the contract. These physical applications include response of structures to various types of random excitation, modification of response due to loading, analysis of fatigue properties, and nonlinear effects. All of these results are discussed in detail in the final report ASD TR 61-123, December 1961.

1.2 OBJECTIVES OF PRESENT CONTRACT

The present contract, "Advanced Concepts of Stochastic Processes and Statistics for Flight Vehicle Vibration Estimation and Measurement," Contract AF33(657)-7459, was sponsored by the Aeronautical Systems Division to extend and continue the work discussed above. This new contract was divided into both theoretical and experimental programs.

On the experimental side, a complete laboratory program was to be carried out to verify the analytical techniques developed for decreasing the amount of data to be gathered, testing fundamental assumptions, and making desired measurements from single experiments and from repeated experiments.

This experimental program was deemed to be of great importance for establishing practical limitations to be attached to previous theoretical work. A major objective was to show how an experimental program should be designed along statistical lines so as to obtain significant information with a limited amount of data. The tests for fundamental assumptions involved new concepts which had never been tested experimentally before. It was expected that further theoretical refinements and improved procedures would be found. It was desired to determine the accuracy of various vibration measurements for a wider range of operating conditions than had been considered elsewhere.

The experimental program was divided into three main areas:

(1) Verification of statistical accuracy of measurements,

(2) Verification of fundamental assumption tests,

(3) Verification of statistical procedures for repeated experiments and sampling techniques.

On the theoretical side, greater emphasis was now to be placed on joint statistical estimation procedures for obtaining area (multiple point) information of a vibration environment as opposed to single point information. Measurable results expected from point to point on an extended structure, as obtained from one or many flights, were to be studied theoretically so as to lead to appropriate mathematical methods for evaluating this information. Practical limitations in these procedures, sampling considerations, and statistical accuracy of the results were important requirements of the analysis.

Another theoretical objective was to investigate effects of nonlinearities on structural vibration responses. In particular, a survey was desired of known technical work devoted to these nonlinear matters so as to expose existing problem areas.

A final theoretical objective was to study characteristics of nonstationary random vibration processes and to try to develop appropriate methods for their analysis. This topic was not studied during the first contract. It was recognized that the whole subject of the nonstationary data analysis was a major unsolved problem, and that considerable effort might be devoted here without achieving significant results. The importance of nonstationary random processes, however, made this topic worthy of special attention.

The theoretical program was divided into four main areas:

(1) Mathematical methods for analyzing nonstationary data.

(2) Statistical procedures for evaluating data from many points.

(3) Survey of methods for predicting the response of nonlinear systems to random vibration.

(4) Further study on sampling techniques with consideration to results from experimental program.

1.3 PERSONNEL OF CONTRACT

The same RW personnel worked on this contract as in the first contract with Dr. J. S. Bendat acting as project manager. He was responsible for promoting mutual awareness and joint participation on different problem areas, and for assigning and reviewing work performed by the other personnel. The theoretical program was of particular concern to him, and his main contributions there were devoted to developing the methods for analyzing nonstationary data which are discussed in this report.

The experimental program was directed by Mr. A. G. Piersol. Mr. Piersol carried out many parts of the experimental program by himself, from initial statistical design to instrument calibration to final evaluation, and he was responsible for formulating several new theoretical improvements to previously suggested tests which made the tests more precise and more practical.

Two other men from RW contributed greatly to the project: Mr. G. H. Klein and Mr. L. D. Enochson. Mr. Klein was responsible for the theoretical section in this report which discusses analysis of variance procedures for evaluating vibration data from many points, as well as for originating some of the basic material discussed under sampling considerations. He conducted also the review study on the response of nonlinear systems to random excitation. Mr. Enochson worked on the statistical design of experiments that were carried out for the experimental program, and assisted in the reduction and interpretation of the collected data. He studied also the statistical problems relating to the analysis of variance procedures.

A small subcontract was given to the Norair Division of Northrop Corporation to provide laboratory facilities for parts of the experimental program. The assistance of Mr. Warren L. Tribble of Norair in performing the experiments was of great help to the success of these experiments.

1.4 SUMMARY OF MAIN RESULTS

1.4.1 Theoretical Program

Significant new theoretical results are developed in Part I of this report for analyzing nonstationary data by appropriate mathematical models and formulas. Starting from first principles, the investigation includes:

* Explanation of differences in analysis required for nonstationary data versus stationary data.
* Formulation of essential parameters required for statistical confidence in nonstationary data measurements.
* Methods for evaluating mean value estimates and mean square value estimates of nonstationary data.
* Formulas for defining correlation functions and spectral density functions of nonstationary data.
* Procedures for determining input-output relations for non-stationary data through linear systems.
* Illustration of these techniques on various examples of physical interest.

Other important theoretical material in Part I, which extends previous work carried out in the first contract as well as complies with the objectives in the present contract, is concerned with:

* Establishment of sampling considerations for flight vehicle vibration problems.
* Evaluation of vibration data from many points on an extended structure as obtained during one or many flights.
* Application of analysis of variance procedures to a wide class of problems.
* Review of various known results on the response of nonlinear systems to random excitation.

1.4.2 Experimental Program

The experimental program is discussed fully in Part II of this report. Significant results of the experimental program include:

* Greater understanding of application areas for previous theoretical work.
* Establishment of new improved theoretical and experimental procedures to test fundamental assumptions of randomness, stationarity, and normality.
* Verification of theoretical statistical uncertainty formulas for experimental measurements of zero crossing estimates, power spectra (mean square) estimates, and amplitude probability density estimates.
* Accuracy of these statistical uncertainty formulas proved experimentally over a wide range of practical center frequencies, bandwidths, and record lengths.
* Demonstration of the importance of the statistical design of experiments to obtain maximum information from a limited amount of data.
* Appreciation for the improvement in statistical conclusions that can be drawn from repeated experiments over single experiments.

1.5 THE TWO PARTS OF THE REPORT

This report is divided into two parts corresponding to the separate theoretical program and experimental program, respectively. During the course of the experimental program, new theoretical results were obtained pertinent to the experimental work which are included with the experimental material. Previous theoretical formulas for the experimental program are summarized with the experimental material. Thus, Part II, Experimental Studies of Random Signal Measurements, is self-contained and may be read completely independent of Part I, Theoretical Studies of Random Signal Estimation.

2. MATHEMATICAL METHODS FOR ESTIMATING MEAN VALUES OF NONSTATIONARY DATA

2.1 INTRODUCTION

Consider a single record $x(t)$ which is a member of an arbitrary random process $\{x(t)\}$ whose statistical properties change with time t. This single record is not required to be representative of any other record except itself. Time averages computed over this single record do not have to agree with corresponding ensemble averages computed over the entire collection of records. Detailed statistical analysis of this single record, therefore, will yield little information for predicting the behavior of the ensemble of records to which it belongs. Such a single record $x(t)$ is said to be <u>nonstationary</u> and the ensemble $\{x(t)\}$ is said to be a <u>nonstationary random process</u>.

This description of nonstationary data fits many physical problems. Examples in the flight vehicle vibration area include data gathered during take-off phases, during transition zones between missile stage separations, and during rapid deceleration landing periods. Shock phenomena and other transient phenomena would also be considered nonstationary.

In order to analyze nonstationary data properly, it is necessary to gather many samples of the data and to compute appropriate ensemble averages over these records. The results will be, in general, a function of the particular times at which these averages are taken, and will also be a function of the number of available records. The accuracy of the quantities being estimated will depend also upon the underlying signal and noise properties at these particular times.

Material in this report for evaluating nonstationary data is developed in five sections. Statistical concepts for measuring mean values of nonstationary data are covered in this section. Section 3 extends this treatment to problems of estimating nonstationary mean square values. This is followed in Section 4 by theoretical material devoted to nonstationary correlation functions and spectral density functions. Section 5 then discusses input-output relations for nonstationary random processes through linear systems, after which a special important physical example is developed in Section 6 to illustrate the various derived formulas.

2.2 MATHEMATICAL MODEL

Consider a random process $f(t)$ which is composed of the sum of a signal process $s(t)$ and an independent noise process $n(t)$. Let $\{f_i(t)\}$, $i = 1, 2, \ldots, N$, $0 \leq t \leq L$, be N samples of $f(t)$. The time t in each $f_i(t)$ is required to be measured from a well-defined origin. The $f_i(t)$ represent an ensemble of functions over which ensemble averages will be taken at fixed values of t. Thus, results will apply to nonstationary data where these ensemble averages change with time. See Figure 2.1.

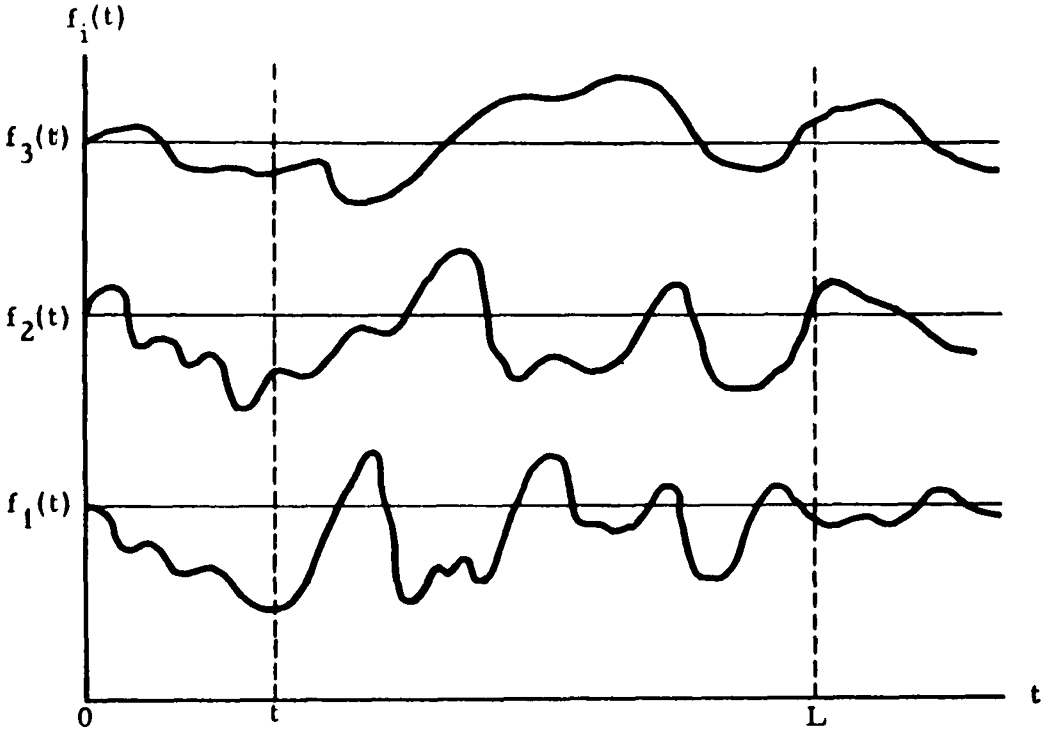

Figure 2.1 Ensemble of Functions

Assume that the noise process $\{n_i(t)\}$ at any time t has an ensemble mean value of zero and a variance of $\sigma_n^2(t)$. Let the signal process $\{s_i(t)\}$ have a population mean value of $\mu(t)$ and a variance of $\sigma_s^2(t)$. Then since $f_i(t) = s_i(t) + n_i(t)$, where $\{s_i(t)\}$ and $\{n_i(t)\}$ are independent, it follows that the ensemble averages, denoted by angular brackets, are

$$\langle f_i(t) \rangle = \langle s_i(t) \rangle = \mu(t) \tag{2. 1}$$

$$\sigma_f^2(t) = \langle f_i^2(t) \rangle - \mu^2(t) = \sigma_s^2(t) + \sigma_n^2(t) \tag{2. 2}$$

For stationary data, these quantities would be independent of time.

The problem at issue is to determine how closely one can estimate $\mu(t)$ by averaging a sufficiently large number of the $f_i(t)$ so as to reduce the interfering noise effects.

At any time t, an average response computer calculates a sample mean value $m(t)$ from a sample of size N as given by

$$m(t) = \frac{1}{N} \sum_{i=1}^{N} f_i(t) = \frac{1}{N} \sum_{i=1}^{N} \left[s_i(t) + n_i(t) \right] \tag{2. 3}$$

The quantity $m(t)$ is a random variable which differs over different choices of the N samples $\{f_i(t)\}$, $i = 1, 2, \ldots, N$. These different choices of the N samples $\{f_i(t)\}$ might be denoted by $\{f_{ik}(t)\}$, $k = 1, 2, \ldots, M$. Then, for a fixed index i, ensemble averages over the index k as M approaches infinity yields

$$\begin{aligned} \langle f_i(t) \rangle &= \lim_{M \to \infty} \frac{1}{M} \sum_{k=1}^{M} f_{ik}(t) = \mu(t) \\ \langle f_i^2(t) \rangle &= \lim_{M \to \infty} \frac{1}{M} \sum_{k=1}^{M} f_{ik}^2(t) \end{aligned} \tag{2. 4}$$

The mean value of the sample mean m(t) over all possible choices of N samples is now

$$\left\langle m(t) \right\rangle = \frac{1}{N} \sum_{i=1}^{N} \left\langle f_i(t) \right\rangle = \frac{1}{N} \sum_{i=1}^{N} \mu(t) = \mu(t) \qquad (2.5)$$

This result is independent of the sample size N, but is a function of time for nonstationary data. Thus the expected value of m(t) as calculated by an average response computer is the desired value $\mu(t)$. In statistical language, m(t) is an unbiased estimate of $\mu(t)$.

The next problem is to estimate the variance associated with this measurement.

The variance of the sample mean is defined by

$$\sigma_m^2(t) = \left\langle m^2(t) \right\rangle - \left(\left\langle m(t) \right\rangle \right)^2 = \left\langle m^2(t) \right\rangle - \mu^2(t) \qquad (2.6)$$

where

$$\left\langle m^2(t) \right\rangle = \frac{1}{N^2} \left\langle \left[\sum_{i=1}^{N} f_i(t) \right]^2 \right\rangle = \frac{1}{N^2} \left[\sum_{i=1}^{N} \left\langle f_i^2(t) \right\rangle + \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \left\langle f_i(t) f_j(t) \right\rangle \right] \qquad (2.7)$$

Further analysis is now governed mainly by the nature of the second summation above. Three cases will be distinguished here according to whether or not $f_i(t)$ and $f_j(t)$ for $i \neq j$ are (1) independent, (2) dependent through a common signal $s_i(t)$ in both $f_i(t)$ and $f_j(t)$ by the condition $\left\langle f_i(t) f_j(t) \right\rangle = \left\langle s_i^2(t) \right\rangle$ for $i \neq j$, (3) correlated in a manner to be defined later.

CASE 1. Independent Samples

If $f_i(t)$ is independent of $f_j(t)$ for all $i \neq j$, then there are $(N^2 - N)$ terms in the second sum where $i \neq j$ of form $\langle f_i(t) f_j(t) \rangle = \langle f_i(t) \rangle \langle f_j(t) \rangle = \mu^2(t)$, while there are N terms in the first sum of form $\langle f_i^2(t) \rangle$. Hence,

$$\langle m^2(t) \rangle = \frac{1}{N^2} \left[N \langle f_i^2(t) \rangle + (N^2 - N) \; \mu^2(t) \right]$$

$$= \frac{1}{N} \left[\langle f_i^2(t) \rangle - \mu^2(t) + N \mu^2(t) \right]$$

$$= \frac{1}{N} \sigma_f^2(t) + \mu^2(t) \tag{2.8}$$

Then,

$$\sigma_m^2(t) = \frac{1}{N} \sigma_f^2(t) = \frac{1}{N} \left[\sigma_s^2(t) + \sigma_n^2(t) \right] \tag{2.9}$$

In words, Eq. (2.9) states that the variance of the sample mean at time t equals the variance of the original random process divided by the sample size. The quantities $\sigma_s^2(t)$ and $\sigma_n^2(t)$ are the variances in the signal and noise, respectively, at time t. For large N, the variance $\sigma_m^2(t)$ approaches zero. Hence, $m(t)$ is a consistent estimate of $\mu(t)$. This result, it should be emphasized, is only for the case when the records $f_i(t) = s_i(t) + n_i(t)$ and $f_j(t) = s_j(t) + n_j(t)$ are statistically independent for $i \neq j$. To be specific, for $i \neq j$, $\langle f_i(t) f_j(t) \rangle$ must be given by $\langle f_i(t) \rangle \langle f_j(t) \rangle$ or one would not satisfy the requirements here. More general formulas to handle situations when $f_i(t)$ and $f_j(t)$ are dependent will be developed later.

CASE 1. Signal-to-Noise Ratios

In terms of signal-to-noise ratios, define

$$\rho_i(t) = \frac{\langle f_i(t) \rangle}{\sigma_f(t)} = \text{input signal-to-noise (amplitude) ratio at time } t \tag{2.10}$$

Define

$$\rho_o(t) = \frac{\langle m(t) \rangle}{\sigma_m(t)} = \text{output signal-to-noise (amplitude) ratio at time } t \tag{2.11}$$

From Eqs. (2.1), (2.2), (2.8), and (2.9), one proves

$$\rho_o(t) = \sqrt{N}\ \rho_i(t) \tag{2.12}$$

showing that for a given input ratio, the output ratio improves directly as the square root of the number N of samples. The relation of $\rho_o(t)$ to the input variance ratio $\sigma_n^2(t)/\sigma_s^2(t)$ is expressed also in Eq. (2.12).

By definition

$$\rho_i(t) = \frac{\mu(t)}{\left[\sigma_s^2(t) + \sigma_n^2(t)\right]^{1/2}} = \frac{\mu(t)/\sigma_s(t)}{\left[1 + \left(\sigma_n^2(t)/\sigma_s^2(t)\right)\right]^{1/2}} \tag{2.13}$$

assuming $\sigma_s(t) \neq 0$. If $\sigma_s(t) = 0$, then the following equations would have to be modified by using $\rho_i(t) = \left[\mu t)/\sigma_n(t)\right]$. Let

$$\frac{\mu(t)}{\sigma_s(t)} = A(t), \text{ a dimensionless quantity} \tag{2.14}$$

A(t) is the ratio of the signal mean value to its standard deviation.

It is clear that if $\mu(t) \equiv 0$, then Eqs. (2.12) to (2.14) all equal zero for all t so that no improvement occurs. Thus, results derived here show that an average response computer will give a zero output when the desired signal has $\mu(t) \equiv 0$. For these situations, a more sophisticated analysis will be required such as, perhaps, the analysis of autocorrelation functions if other signal properties are desired.

Assuming $\mu(t) \neq 0$, substitution of Eqs. (2.13) and (2.14) into Eq. (2.12) yields for the output signal-to-noise ratio

$$\rho_o(t) = \frac{A(t)\sqrt{N}}{\left[1 + \left|\sigma_n^2(t)/\sigma_s^2(t)\right|\right]^{1/2}} \tag{2.15}$$

$$\approx \frac{A(t)\sqrt{N}}{\left[\sigma_n(t)/\sigma_s(t)\right]} = \frac{\mu(t)\sqrt{N}}{\sigma_n(t)} \quad \text{if } \sigma_n^2(t) >> \sigma_s^2(t) \tag{2.16}$$

These last two equations show that the output signal-to-noise ratio is always directly proportional to $\sqrt{N}$, and inversely proportional to $\sigma_n(t)/\sigma_s(t)$ when the noise variance $\sigma_n^2(t)$ is large compared to the signal variance $\sigma_s^2(t)$. In order to compensate for a large expected input ratio of $\sigma_n(t)/\sigma_s(t)$, it is required to increase N accordingly.

For example, suppose $(\sigma_n/\sigma_s) = 10$ at some particular time and one desires $(\rho_o/A) = 1$. Then, it is necessary that $N = 100$ samples. For a case when $N = 25$ samples, if $(\sigma_n/\sigma_s) = 10$, it follows that $(\rho_o/A) = 0.5$. In order to have $(\rho_o/A) = 1$ when $N = 25$, the maximum allowable ratio for (σ_n/σ_s) would be $\sqrt{24} \approx 4.90$.

Equation (2.15) is one of the important results of this section. Table 2.1, plotted in Figure 2.2, shows the quantity $\rho_o(t)/A(t)$ as a function of N for given values of $\sigma_n(t)/\sigma_s(t)$. In actual practice, the quantity $A(t)$ will in many situations be larger than unity. The output ratio $\rho_o(t)$ itself will be $A(t)$ times the values shown on the vertical scale.

σ_n/σ_s	N											
	10	20	30	40	50	60	70	80	90	100	110	120
0.5	2.8	4.0	4.9	5.6	6.3	6.9	7.4	7.9	8.4	8.9	9.4	9.8
1.0	2.3	3.2	3.9	4.5	5.0	5.5	5.9	6.3	6.7	7.1	7.4	7.7
2.0	1.4	2.0	2.5	2.9	3.2	3.5	3.8	4.1	4.3	4.5	4.7	4.9
5.0	0.63	0.88	1.08	1.24	1.40	1.52	1.65	1.75	1.87	1.97	2.06	2.15
10.0	0.32	0.45	0.55	0.63	0.71	0.77	0.84	0.89	0.95	.995	1.04	1.09

Table 2.1 Data for Figure 2.2 from Eq. (2.15)

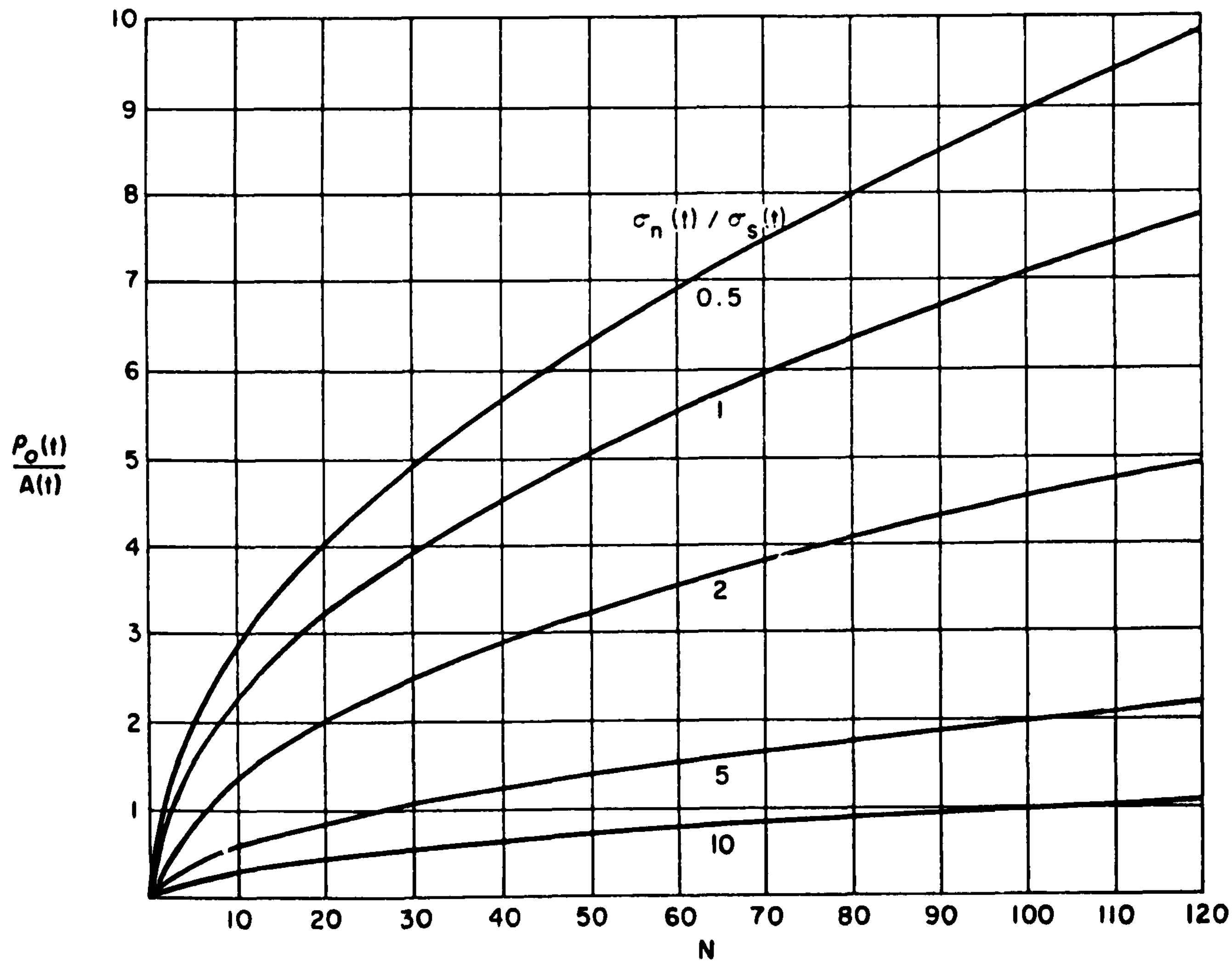

Figure 2.2 Plot of $\rho_o(t)/A(t)$ as function of N for various $\sigma_n(t)/\sigma_s(t)$ from Eq.(2.15)

CASE 1. Confidence Limits

A knowledge of the mean value and variance for the random variable m(t) at any time t enables one to answer questions concerning the range of the results at any time t without knowing the exact probability distribution function for m(t). From the Tchebycheff Inequality, which applies to arbitrary general situations, one may state with 89% confidence, for example, that an observed measurement for m(t) lies inside the range $\left[\mu(t) - 3\sigma_m(t),\ \mu(t) + 3\sigma_m(t)\right]$. In equation form, for any constant λ, the Tchebycheff Inequality is

$$\text{Prob}\left[\left|m(t) - \mu(t)\right| \geqq \lambda\sigma_m(t)\right] \leqq \frac{1}{\lambda^2} \qquad (2.17)$$

Thus, for $\lambda = 3$, the probability is at most (1/9), giving the above 89% confidence limits. See Figure 2.3.

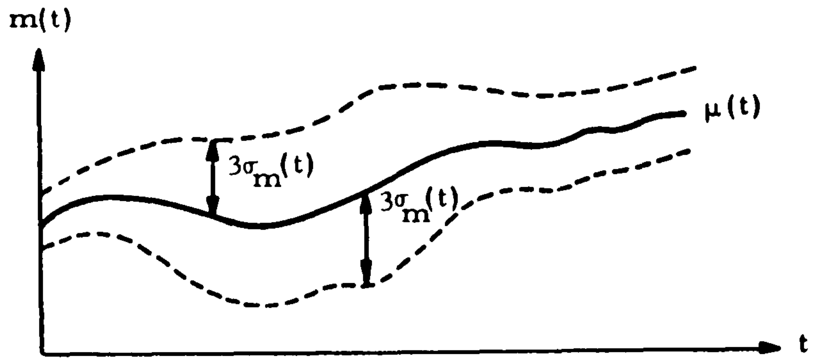

Figure 2.3 89% Confidence Limits for Arbitrary Distribution

A stronger statement can be made if one can justify an assumption that m(t) follows a normal (Gaussian) distribution at any value of t. For this special case, a 95% confidence band is given by the range $\left[\mu(t) - 2\sigma_m(t),\ \mu(t) + 2\sigma_m(t)\right]$. Thus, an observed measurement for m(t) in the Gaussian case yields a greater confidence of being close to the theoretical mean value $\mu(t)$ than before.

Consider the range for m(t) as given by

$$|m(t) - \mu(t)| \leqq k\sigma_m(t) \tag{2.18}$$

where k is a constant. Solving for m(t) yields the two extreme range values

$$m(t) = \mu(t) \pm k\sigma_m(t) \tag{2.19}$$

Upon substitution of the general expression for $\sigma_m(t)$ from **Eq.** (2.9), one finds

$$\begin{aligned} m(t) &= \mu(t) \pm \lambda \frac{\sqrt{\sigma_s^2(t) + \sigma_n^2(t)}}{\sqrt{N}} \\ &= \mu(t) \pm \lambda\sigma_s(t) \frac{\left[1 + \left|\sigma_n^2(t)/\sigma_s^2(t)\right|\right]^{1/2}}{\sqrt{N}} \end{aligned} \tag{2.20}$$

Hence, by increasing N, one may guarantee that m(t) will fall close to $\mu(t)$, regardless of the magnitude of the other variables.

Equation (2.20) can be put into another equivalent form by using Eq(2.15). This yields a new interesting expression

$$m(t) = \mu(t) \pm \lambda \frac{\mu(t)}{\rho_o(t)} \tag{2.21}$$

Now, on solving for $\mu(t)$, one derives

$$\mu(t) = \frac{m(t)}{1 \pm \left[\lambda/\rho_o(t)\right]} \qquad \text{for} \qquad \rho_o(t) \gg k \tag{2.22}$$

which shows how range values for m(t) are related to the output signal-to-noise (amplitude) ratio $\rho_o(t)$, and to the constant λ determining the confidence limits. Clearly, for good results, $\rho_o(t)$ should be as large as possible relative to λ in order for m(t) to approximate $\mu(t)$.

Equation (2.22), like Eq. (2.15), is a major result of this section. From Eq(2.22), for example, if $\lambda = 3$ and $\rho_o(t) = 6$, then for an unspecified probability distribution function associated with m(t), the Techebycheff

Inequality yields 89% confidence that $\mu(t)$ lies in the range bounded by

$$\mu(t) = \frac{m(t)}{1 \pm (1/2)} = \left[\frac{2}{3} m(t), 2m(t)\right]$$

Thus, one measures $m(t)$ and then applies the above bounds to estimate $\mu(t)$. If a Gaussian distribution is assumed, then for $\lambda = 2$ and $\rho_o(t) = 6$, one can state with 95% confidence that $\mu(t)$ lies in the smaller range bounded by

$$\mu(t) = \frac{m(t)}{1 \pm (1/3)} = \left[\frac{3}{4} m(t), \frac{3}{2} m(t)\right]$$

Figure 2.4 based on Eq. (2.22) is a plot of the ratio $\mu(t)/m(t)$ as a function of $\rho_o(t)$. Two cases are considered. Case 1 applies to arbitrary probability distributions and sets $\lambda = 3$, corresponding to an 89% confidence band as given by the Tchebycheff Inequality. Case 2 applies to a Gaussian probability distribution and sets $\lambda = 2$, corresponding to a 95% confidence band. The lower and upper limits used for Figure 2.4 are shown in Table 2.2.

Substitution of Eq. (2.12) into (2.22) yields a useful general result

$$\mu(t) = \frac{m(t)}{1 \pm \left[\lambda/\sqrt{N}\ \rho_i(t)\right]} \qquad \text{for} \qquad \sqrt{N}\ \rho_i(t) \succ \lambda \tag{2.23}$$

which shows how, for given values of λ and $\rho_i(t)$, one can make $m(t)$ fall arbitrary close to $\mu(t)$ by proper choice of N. Note that if $\rho_i(t)$ is assumed to be unity, then $\rho_o(t) = \sqrt{N}$. For this case, the numbers shown in Table 2.2 and Fig. 2.4 can be interpreted quite simply in terms of sample size N. For example, $\rho_o(t) = 10$ would correspond to $N = 100$ and $\rho_o(t) = 20$ corresponds to $N = 400$.

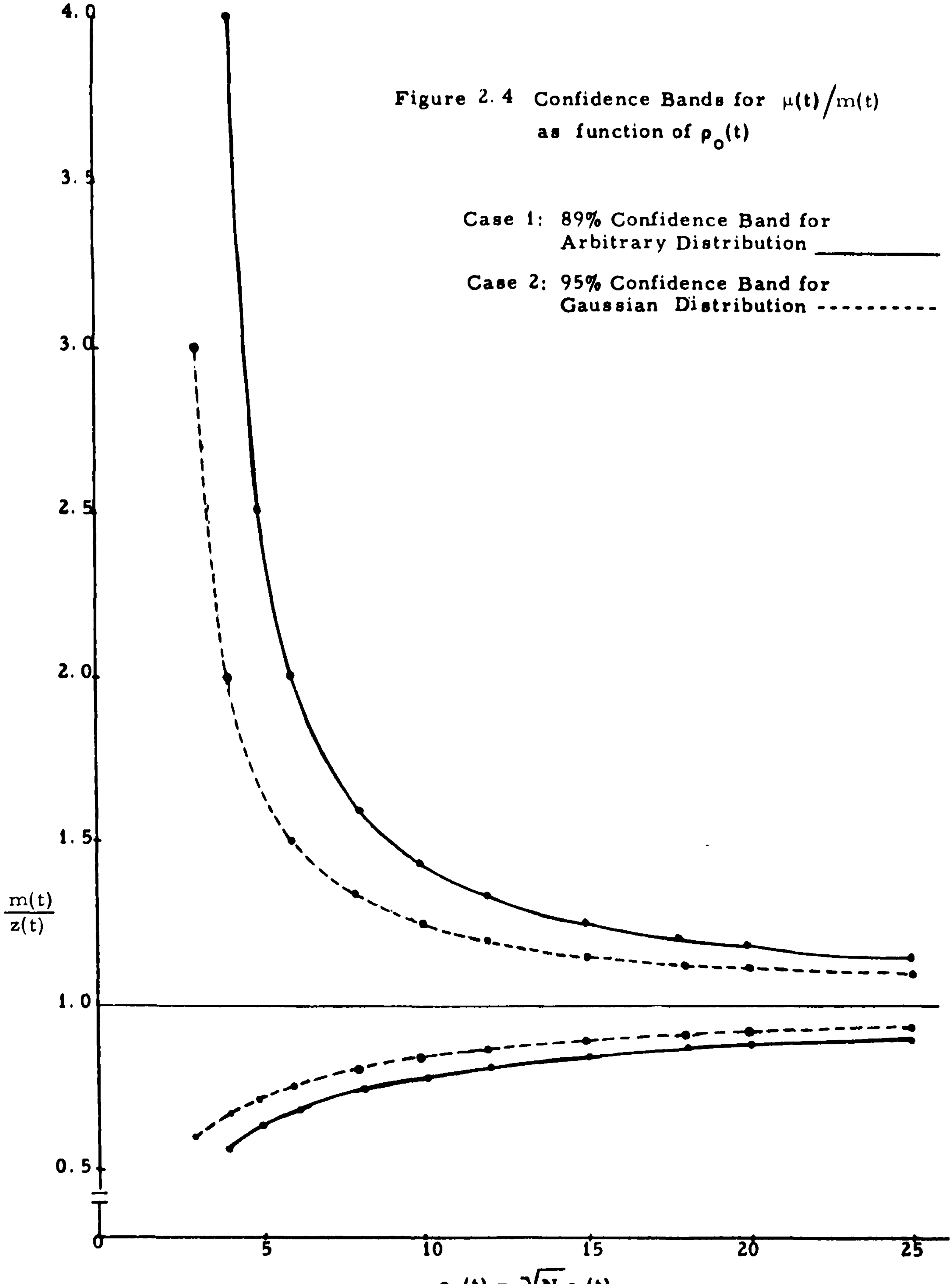
Figure 2.4 Confidence Bands for $\mu(t)/m(t)$ as function of $\rho_o(t)$
Case 1: 89% Confidence Band for Arbitrary Distribution
Case 2: 95% Confidence Band for Gaussian Distribution
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0
$\frac{m(t)}{z(t)}$
5
10
15
20
25
$\rho_o(t) = \sqrt{N}\,\rho_i(t)$

$\rho_o(t)$	Case 1		Case 2	
	Lower Limit	Upper Limit	Lower Limit	Upper Limit
2	---	---	0.50	∞
3	0.50	∞	0.60	3.00
4	0.57	4.00	0.67	2.00
5	0.63	2.50	0.71	1.67
6	0.67	2.00	0.75	1.50
8	0.73	1.60	0.80	1.33
10	0.77	1.43	0.83	1.25
12	0.80	1.33	0.86	1.20
15	0.83	1.25	0.88	1.15
18	0.86	1.20	0.90	1.12
20	0.87	1.18	0.91	1.11
25	0.89	1.14	0.93	1.09

Table 2.2 Data for Figure 2.4

Equation (2.23) indicates also how a sample size N_1 required in arbitrary non-Gaussian cases is related to the sample size N_2 required in Gaussian cases. To illustrate this point, for an arbitrary distribution and arbitrary input ratio $\rho_i(t)$, the parameter $\lambda_1 = 3$ corresponds to an 89% confidence band. For the same input ratio and the same 89% confidence band, a Gaussian distribution has associated with it the parameter $\lambda_2 = 1.60$. From Eq. (2.23), the corresponding sample sizes N_1, N_2 must satisfy

$$\frac{\lambda}{\sqrt{N}} = \frac{3}{\sqrt{N_1}} = \frac{1.6}{\sqrt{N_2}} \quad \text{or} \quad N_1 = 3.5\,N_2$$

Thus, for example, an acceptable sample size of $N_2 = 30$ for an 89% confidence band with a Gaussian distribution would need to be

increased to $N_1 = 105$ for an unspecified distribution in order to obtain the same 89% confidence band. Consider a different case when $\lambda_1 = 2$, corresponding to a 75% confidence band from the Tchebycheff Inequality. The associated Gaussian parameter value is $\lambda_2 = 1.15$. This leads to

$$\frac{2}{\sqrt{N_1}} = \frac{1.15}{\sqrt{N_2}} \quad \text{or} \quad N_1 = 3N_2$$

Now, $N_2 = 30$ would need to be increased only to $N_1 = 90$ to obtain a 75% confidence band.

Related sample sizes for N_1 and N_2 as a function of confidence band are summarized in Table 2.3. The sample size N_1 is the sample size for an unknown distribution while N_2 is the sample size for a Gaussian distribution

Confidence Band	N_1 / N_2
75%	3.0
89%	3.5
92%	4.1
95%	5.2
98%	9.2
99%	15.0

N_1 = sample size required for unknown distribution

N_2 = sample size required for Gaussian distribution

Table 2.3 Related Sample Sizes for Unknown Versus Gaussian Distribution

CASE 2. Dependent Samples

For this case, assume that $f_i(t)$ and $f_j(t)$ have the form

$$f_i(t) = s_i(t) + n_i(t)$$
$$f_j(t) = s_i(t) + n_j(t) \tag{2.24}$$

where $s_i(t)$ is a common random signal in both $f_i(t)$ and $f_j(t)$, while $n_i(t)$ and $n_j(t)$ are noise terms independent of each other and of $s_i(t)$. As in Case 1, the ensemble averages are given by

$$\left\langle f_i(t)\right\rangle = \left\langle s_i(t)\right\rangle = \mu(t)$$
$$\left\langle f_i^2(t)\right\rangle = \left\langle s_i^2(t)\right\rangle + \left\langle n_i^2(t)\right\rangle = \sigma_s^2(t) + \mu^2(t) + \sigma_n^2(t) \tag{2.25}$$
$$\sigma_f^2(t) = \sigma_s^2(t) + \sigma_n^2(t)$$

The change from Case 1 is in the term

$$\left\langle f_i(t)\, f_j(t)\right\rangle = \left\langle s_i^2(t)\right\rangle \quad \text{for} \quad i \neq j \tag{2.26}$$

Previously it was assumed that $f_i(t)$ and $f_j(t)$ were independent for $i \neq j$, so that $\left\langle f_i(t)\, f_j(t)\right\rangle = \mu^2(t)$.

Return now to Eq.(2.7). Substitution from Eqs.(2.25) and (2.26) into Eq.(2.7) yields

$$\left\langle m^2(t)\right\rangle = \frac{1}{N^2}\left[N\left(\left\langle s_i^2(t)\right\rangle + \left\langle n_i^2(t)\right\rangle\right) + (N^2 - N)\left\langle s_i^2(t)\right\rangle\right]$$
$$= \left\langle s_i^2(t)\right\rangle + \frac{1}{N}\left\langle n_i^2(t)\right\rangle \tag{2.27}$$

From Eq.(2.5) and Eq.(2.26), the ensemble average

$$\left\langle m(t)\right\rangle = \left\langle s_i(t)\right\rangle = \mu(t) \tag{2.28}$$

Thus, $\sigma_m^2(t)$ becomes

$$\sigma_m^2(t) = \sigma_s^2(t) + \frac{\sigma_n^2(t)}{N} \tag{2.29}$$

which is quite different from Eq. (2.9), the result derived for Case 1.

In words, Eq. (2.29) states that the variance of the sample mean at time t equals the variance of the signal portion at time t plus (1/N) times the variance of the noise portion at time t. For large N, the variance $\sigma_m^2(t)$ approaches $\sigma_s^2(t)$ which is not required to be zero. Hence m(t) is no longer a consistent estimate of $\mu(t)$. Eq. (2.29) shows that m(t) is still, however, an unbiased estimate of $\mu(t)$.

Example 1: Sine Wave process with dependent amplitudes.

$$s_i(t) = a_i \cos(\omega t + \phi) \quad ; \quad a_i \text{ variable}, \ \langle a_i a_j \rangle = \langle a_i^2 \rangle \neq \left(\langle a_i \rangle \right)^2$$

$$\langle s_i(t) \rangle = \mu(t) = \langle a_i \rangle \cos(\omega t + \phi)$$

$$\langle s_i^2(t) \rangle = \langle a_i^2 \rangle \cos^2(\omega t + \phi)$$

$$\sigma_s^2(t) = \sigma_a^2 \cos^2(\omega t + \phi) \quad \text{where} \quad \sigma_a^2 = \langle a_i^2 \rangle - \left(\langle a_i \rangle \right)^2$$

$$\sigma_m^2(t) \rightarrow \sigma_s^2(t) \qquad \text{as} \qquad N \rightarrow \infty$$

Here, one is not able to uncover $\mu(t)$ as closely as desired since $\sigma_m^2(t)$ does not approach zero for large N. If the amplitudes were independent, $\sigma_m^2(t)$ would approach zero.

Example 2: Sine Wave process with variable phase angles.

$$s_i(t) = a \cos(\omega t + \phi_i) \quad ; \quad a \text{ constant}, \ \phi_i \text{ variable}$$

$$\langle s_i(t) \rangle = \mu(t) \equiv 0 \quad \text{for all } t$$

$$\langle s_i^2(t) \rangle = (a^2/2)$$

$$\sigma_s^2(t) = (a^2/2)$$

Since $\mu(t) \equiv 0$ here, an average response computer will not uncover any properties of this signal except for this fact.

CASE 2. Signal-to-Noise Ratios

For the following discussion, assume that $\mu(t) \neq 0$. From Eqs.(2.10) and (2.11), one finds for Case 2 that the input ratio

$$\rho_i(t) = \frac{\langle f_i(t) \rangle}{\sigma_f(t)} = \frac{\mu(t)}{\left[\sigma_s^2(t) + \sigma_n^2(t)\right]^{1/2}} \tag{2.30}$$

while the output ratio

$$\rho_o(t) = \frac{\langle m(t) \rangle}{\sigma_m(t)} = \frac{\mu(t)}{\left[\sigma_s^2(t) + (\sigma_n^2(t)/N)\right]^{1/2}} \tag{2.31}$$

Thus,

$$\rho_o(t) = \frac{\rho_i(t)\left[\sigma_s^2(t) + \sigma_n^2(t)\right]^{1/2}}{\left[\sigma_s^2(t) + (\sigma_n^2(t)/N)\right]^{1/2}} \tag{2.32}$$

Eq(2.33) proves that $\rho_o(t) > \rho_i(t)$ for all $N > 1$ since $\sigma_n^2(t) > \left[\sigma_n^2(t)/N\right]$, assuming that $\sigma_n^2(t) \neq 0$.

As in Eq(2.14), assuming $\mu(t)$ and $\sigma_s(t)$ to be different from zero, let

$$A(t) = \frac{\mu(t)}{\sigma_s(t)} \tag{2.33}$$

In terms of A(t), Eqs. (2.30) and (2.31) can be written

$$\rho_i(t) = \frac{A(t)}{\left[1 + (\sigma_n^2(t)/\sigma_s^2(t)\right]^{1/2}} \tag{2.34}$$

$$\rho_o(t) = \frac{A(t)\sqrt{N}}{\left[N + (\sigma_n^2(t)/\sigma_s^2(t)\right]^{1/2}} \tag{2.35}$$

The latter important result should be compared with Eq(2.15). Note that in Eq(2.15), the denominator contains "1" instead of "N".

Consider a few special properties of Eq. (2.35). Always

$$\frac{\rho_o(t)}{A(t)} = \frac{\sqrt{N}}{\left[N + (\sigma_n^2(t)/\sigma_s^2(t))\right]^{1/2}} \leqq 1 \tag{2.36}$$

since $(\sigma_n^2(t)/\sigma_s^2(t)) \geqq 0$. For large N,

$$\rho_o(t) \longrightarrow A(t) \quad \text{as} \quad N \longrightarrow \infty \tag{2.37}$$

In order for $\rho_o(t) \geqq 1$, it is necessary that

$$A(t) \geqq \frac{\left[N + (\sigma_n^2(t)/\sigma_s^2(t))\right]^{1/2}}{\sqrt{N}} \tag{2.38}$$

To consider some numerical examples, suppose $(\sigma_n/\sigma_s) = 10$ at some particular time and $N = 100$. Then $\rho_o = 0.7A$. In order to have $\rho_o = 0.5A$ when $N = 25$, the maximum allowable ratio for (σ_n/σ_s) would be $\sqrt{75} \approx 8.66$. For this latter example, if $(\sigma_n/\sigma_s) = \sqrt{75}$ and $A = 10$, then $\rho_i \approx 1.15$ and $\rho_o = 5$.

Table 2.4, shows the quantity $\rho_o(t)/A(t)$ as a function of N for given values of $\sigma_n(t)/\sigma_s(t)$. The actual output ratio $\rho_o(t)$ will be $A(t)$ times the value shown in the table. It is instructive to compare Table 2.4 with Table 2.1 to appreciate the difference between Case 1 and Case 2.

σ_n/σ_s	N									
	10	20	30	40	50	60	70	80	90	100
0.5	0.98	0.99	0.99	0.99+	0.99+	0.99+	0.99+	0.99+	0.99+	0.99+
1.0	0.95	0.97	0.98	0.99	0.99	0.99	0.99+	0.99+	0.99+	0.99+
2.0	0.85	0.92	0.94	0.95	0.96	0.97	0.97	0.98	0.98	0.98
5.0	0.54	0.67	0.74	0.78	0.82	0.84	0.86	0.88	0.89	0.90
10.0	0.30	0.41	0.48	0.54	0.58	0.61	0.64	0.67	0.69	0.71

Table 2.4 Value of $\rho_o(t)/A(t)$ as function of N for various $(\sigma_n(t)/\sigma_s(t))$ from Eq. (2.36).

CASE 2. Confidence Limits

Confidence limits for Case 2 can be calculated by using Eq.(2.19) with the new $\sigma_m(t)$ from Eq.(2.29). This yields

$$
\begin{aligned}
m(t) &= \mu(t) \pm \lambda \left[\sigma_s^2(t) + (\sigma_n^2(t) / N)\right]^{1/2} \\
&= \mu(t) \pm \lambda\, \sigma_s(t) \frac{\left[N + (\sigma_n^2(t)/\sigma_s^2(t)\right]^{1/2}}{\sqrt{N}} \qquad (2.39) \\
&= \mu(t) \pm \lambda \frac{\mu(t)}{\rho_o(t)}
\end{aligned}
$$

upon substitution from Eqs. (2.33) and (2.35). Hence

$$
\mu(t) = \frac{m(t)}{1 \pm \left[\lambda/\rho_o(t)\right]} \qquad \text{for} \qquad \rho_o(t) > \lambda \qquad (2.40)
$$

Eq.(2.40) is precisely the same form as Eq. (2.22), and like Eq.(2.22) is a major result of this investigation.

Further analysis of Eq.(2.40) would proceed exactly as was done in Case 1. All results derived there, including Figure 2.4, Table 2.2, and Table 2.3 apply here without exception. In using Figure 2.4, for example, the only precaution is to be sure that the output ratio for $\rho_o(t)$ is given by its correct form for Case 2, namely, Eq.(2.35) instead of Eq.(2.15) which holds for Case 1.

CASE 3. Correlated Samples

Consider now a general situation where

$$f_i(t) = s_i(t) + n_i(t)$$
$$f_j(t) = s_j(t) + n_j(t) \tag{2.41}$$

such that the ensemble average is defined by

$$\langle f_i(t)\, f_j(t)\rangle = \langle s_i(t)\, s_j(t)\rangle = R_{ss}(k, t) \quad \text{where} \quad k = j - i \tag{2.42}$$

The quantity $R_{ss}(k, t)$ is a nonstationary spatial cross-correlation function at time t between the ith record $s_i(t)$ and the jth record $s_j(t)$ such that $j - i = k$. It follows from the above definition by interchanging i and j that

$$R_{ss}(k, t) = R_{ss}(-k, t) \tag{2.43}$$

To illustrate, this quantity, for N records, one would set

$$R_{ss}(0, t) = \langle s_i^2(t)\rangle = \sigma_s^2(t) + \mu^2(t) \approx \frac{s_1^2(t) + s_2^2(t) + \ldots + s_N^2(t)}{N}$$

$$R_{ss}(1, t) = \langle s_i(t)\, s_{i+1}(t)\rangle \approx \frac{s_1(t)s_2(t) + s_2(t)s_3(t) + \ldots + s_{N-1}(t)s_N(t)}{N-1}$$

$$R_{ss}(2, t) = \langle s_i(t)s_{i+2}(t)\rangle \approx \frac{s_1(t)s_3(t) + s_2(t)s_4(t) + \ldots + s_{N-2}(t)s_N(t)}{N-2}$$

and so forth. The true values are obtained as N becomes large.

If the records have been collected at regularly space time intervals L apart, and if t is measured from a well defined origin in each case, then $R_{ss}(k, t)$ can be interpreted as the correlation existing at time t

between records which are kL apart. Thus, $R_{ss}(k,t)$ can be used to define adaptive changes (such as structural fatigue effects or physiological learning patterns) in the records as a function of repeated stimuli which produce the records. Refer to Figure 2.1 and consider $f_1(t)$ to have a time scale $0 \leq t \leq L$, consider $f_2(t)$ to have a time scale $L \leq t \leq 2L$, and so forth.

The precise form of $R_{ss}(k,t)$ determines whether one has a Case 1 situation, a Case 2 situation, or something in between. To be specific, it will be shown shortly that

(1) $R_{ss}(k,t) = \mu^2(t)$ for all $k \geq 1$ yields Case 1

(2) $R_{ss}(k,t) = R_{ss}(0,t)$ for all $k \geq 1$ yields Case 2

For the subsequent discussion, assume as before that $n_i(t)$ and $n_j(t)$ are independent of each other and of $s_i(t)$ and $s_j(t)$. As before, the ensemble average

$$\langle f_i^2(t) \rangle = \langle s_i^2(t) \rangle + \langle n_i^2(t) \rangle = R_{ss}(0,t) + \sigma_n^2(t) \tag{2.44}$$

Consider now Eq.(2.7) and the value of $\langle m^2(t) \rangle$ for this case. Equation (2.7) states that

$$\langle m^2(t) \rangle = \frac{1}{N^2}\left[\sum_{i=1}^{N} \langle f_i^2(t) \rangle + \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \langle f_i(t)\, f_j(t) \rangle\right] \tag{2.45}$$

Substitution from Eqs. (2.44) and (2.42) yields

$$\langle m^2(t) \rangle = \frac{1}{N^2}\left[N\left(R_{ss}(0,t) + \sigma_n^2(t)\right) + \sum_{\substack{i,j=1 \\ k=j-i\neq 0}}^{N} R_{ss}(k,t)\right] \tag{2.46}$$

The problem of concern is the interpretation of the double sum over i and j. The index $k = j - i$ takes on values $1, 2, \ldots, N-1$ Altogether, there are $N^2 - N$ terms. Since $R_{ss}(k, t) = R_{ss}(-k, t)$, the $(N^2 - N)$ terms in this sum can be arranged so that there are two terms where $k = N - 1$ of form $R_{ss}(N - 1, t)$, four terms where $k = N - 2$ of form $R_{ss}(N - 2, t), \ldots,$ and $2(N - 1)$ terms where $k = 1$ of form $R_{ss}(1, t)$. Thus, one derives the simplified expression

$$\sum_{\substack{i, j=1 \\ k=j-i\neq 0}}^{N} R_{ss}(k, t) = 2 \sum_{k=1}^{N-1} (N - k) R_{ss}(k, t) \tag{2.47}$$

As a check, note that the sum

$$2 \sum_{k=1}^{N-1} (N - k) = (N^2 - N) \tag{2.48}$$

Equation (2.47) is a key analytical result. Substitution of Eq. (2.47) into Eq. (2.46) shows

$$\left\langle m^2(t) \right\rangle = \frac{1}{N} \left[R_{ss}(0, t) + \sigma_n^2(t) \right] + \frac{2}{N^2} \sum_{k=1}^{N-1} (N - k) R_{ss}(k, t) \tag{2.49}$$

Finally, since $R_{ss}(0, t) = \sigma_s^2(t) + \mu^2(t)$, with the aid of Eq. (2.48), the variance $\sigma_m^2(t) = \left\langle m^2(t) \right\rangle - \mu^2(t)$ becomes

$$\sigma_m^2(t) = \frac{\sigma_s^2(t) + \sigma_n^2(t)}{N} + \frac{2}{N^2} \sum_{k=1}^{N-1} (N - k) \left[R_{ss}(k, t) - \mu^2(t) \right] \tag{2.50}$$

Equation (2.50) is a major result of this investigation. From it may be derived confidence statements and curves similar to those found previously, where Eq. (2.50) takes the place of Eq. (2.9) or Eq. (2.29).

Two cases of Eq. (2.50) are worthy of special mention

(A) $$R_{ss}(k, t) = \mu^2(t) \quad \text{for all } k > 1 \tag{2.51}$$

From Eq. (2.50), it follows that

$$\sigma_m^2(t) = \frac{\sigma_s^2(t) + \sigma_n^2(t)}{N} \tag{2.52}$$

which is precisely Eq. (2.9) of Case 1. This situation assumes no dependence between the signals in the different records.

(B) $$R_{ss}(k, t) = R_{ss}(0, t) = \sigma_s^2(t) + \mu^2(t) \quad \text{for all } k > 1 \tag{2.53}$$

From Eqs. (2.50) and (2.48), it follows that

$$\sigma_m^2(t) = \sigma_s^2(t) + \frac{\sigma_n^2(t)}{N} \tag{2.54}$$

which is precisely Eq. (2.29) of Case 2. This situation assumes complete dependence between the signals in the different records.

For physical situations where neither Case 1 nor Case 2 apply, Eq. (2.50) gives a more general expression to analyze nonstationary problems where a partial correlation exists between signals in the different records.

CASE 3. Exponential Correlation Function

An exponential form for the nonstationary cross-correlation function $R_{ss}(k, t)$ will now be assumed so as to obtain some quantitative results to characterize different degrees of correlation. This particular form is known to fit many physical situations and it is therefore deemed appropriate for consideration here. To be specific, it will be assumed that

$$R_s(k, t) = \mu^2(t) + \sigma_s^2(t) e^{-|k|c} \tag{2.55}$$

where k and c are positive constants. Observe that an extremely large value for the correlation coefficient c corresponds to Case 1 while an extremely small value for c corresponds to Case 2. Equation (2.50) now becomes

$$\sigma_m^2(t) = \frac{\sigma_s^2(t) + \sigma_n^2(t)}{N} + \frac{2\sigma_s^2(t)}{N^2} \sum_{k=1}^{N-1} (N - k) e^{-kc} \tag{2.56}$$

To evaluate the sum above, let

$$f(c) = \sum_{k=1}^{N-1} e^{-kc} = \frac{1 - e^{-(N-1)c}}{e^c - 1} \tag{2.57}$$

Then

$$f'(c) = - \sum_{k=1}^{N-1} k e^{-kc} = \frac{N e^{-(N-2)c} - (N - 1) e^{-(N-1)c} - e^c}{(e^c - 1)^2} \tag{2.58}$$

Now

$$F(c) = \sum_{k=1}^{N-1} (N - k) e^{-kc} = N f(c) + f'(c)$$

$$= \frac{(N - 1) e^c - N + e^{-(N-1)c}}{(e^c - 1)^2} \tag{2.59}$$

Substitution into Eq.(2.56) gives

$$\sigma_m^2(t) = \frac{\sigma_s^2(t) + \sigma_n^2(t)}{N} + \frac{2\sigma_s^2(t)}{N}\left[\frac{(N-1)e^c - N + e^{-(N-1)c}}{(e^c - 1)^2}\right] \quad (2.60)$$

Equation (2.60) can be used to generate a set of curves for different values of N and c. For a given value of N, and for known values of $\sigma_s^2(t)$ and $\sigma_n^2(t)$, actual experimental results would enable one to estimate the correlation coefficient c.

2.3 CONCLUSION

Results obtained in this section may be applied in many ways. These results are valid for nonstationary data since all averages are ensemble averages and the various statistical quantities involved are allowed to change with time. Three cases are distinguished according to whether or not the sample records are (1) independent, (2) dependent, (3) correlated. If input signal and noise variances are known, and if the input signal-to-noise ratio is known, then one may calculate the output signal-to-noise as a function of the sample size. The same formulas enable one to decide in advance how large the sample size should be to achieve a desired output ratio. Knowledge of the output ratio permits definite confidence bands to be associated with measurements from an average response computer.

Working backwards from the derived equations, if one can estimate the output ratio for a known sample size, then one can infer properties about the input ratio, and, thus, about the underlying input signal and noise variances. Similarly, the ability to estimate the confidence band associated with average response computer measurements enables one to estimate the output ratio. Changes in sample size are tabulated to achieve a given confidence band for measurements following an unknown arbitrary distribution, as compared to measurements following a Gaussian distribution.

Estimates of an underlying noise variance alone can sometimes be obtained experimentally when no signal is present, and of combined signal and noise variances when the signal is present. This variance information,

together with a corresponding mean value estimate would provide data for estimating an input signal-to-noise ratio.

The Case 3 situation of correlated samples includes Cases 1 and 2 as special cases. This case enables one to analyze adaptive changes in records as a function of repeated stimuli. A quantitative technique is described, using an exponential correlation function, which may help to characterize different physical situations.

3. ESTIMATION OF NONSTATIONARY MEAN SQUARE VALUES

3.1 INTRODUCTION

The previous section described methods for analyzing the accuracy to be associated with average response (mean value) computations on non-stationary data. The present section continues this line of analysis so as to determine the accuracy to be associated with mean square computations on nonstationary data. One reason for the interest in mean square values is that it describes the "power" contained in the data at any time and thus shows how the power change with time. Another reason is that the mean value is commonly zero when analyzing random phenomena so that one is forced to consider higher moments such as mean square values in order to describe the data.

For the present treatment, it will be assumed that one has available a set of N stationary or nonstationary records $\{f_i(t)\}$, $i = 1, 2, \ldots, N$. This set of records is assumed to have a well-defined time origin in each case, such as the closing of a switch, the opening of a door, the start of an impact, etc. A mean square computer takes each sample record $f_i(t)$, squares it to obtain $f_i^2(t)$, and then averages the set of N different $f_i^2(t)$. This provides an estimate of the true mean square value which would be obtained if one had available an infinitely large collection of records. The problem of this section is to determine how closely one can estimate the true mean square value.

At any time t, a mean square computer calculates a sample mean square value from a set of records $\{f_i(t)\}$ of size N as given by

$$v(t) = \frac{1}{N} \sum_{i=1}^{N} f_i^2(t) \tag{3.1}$$

The sample records $\{f_i(t)\}$ will be assumed to be additive mixtures of independent signals $\{s_i(t)\}$ and noise $\{n_i(t)\}$. Pairs of records $f_i(t)$ and $f_j(t)$ where $i \neq j$ will separate into three cases according to whether or not they are (1) independent, (2) dependent through a common signal $s_i(t)$ in both $f_i(t)$ and $f_j(t)$, (3) correlated in a manner to be defined later. These three cases are precisely the same three cases as were considered previously when analyzing the accuracy to be associated with average response computers for nonstationary data.

The quantity $v(t)$ is an unbiased estimate of the population mean square value $\mu_v(t)$ over $\{f_i(t)\}$ at any time t since the ensemble average over a set of estimates $\{v(t)\}$ is

$$\mu_v(t) = \langle v(t) \rangle = \frac{1}{N} \sum_{i=1}^{N} \langle f_i^2(t) \rangle = \langle f_i^2(t) \rangle = \langle s_i^2(t) \rangle + \langle n_i^2(t) \rangle \quad (3.2)$$

It should be noted here that $v(t)$ is an estimate of the mean square value of the signal plus noise. A cross-correlation procedure is required to estimate the mean square value of the signal alone.

The main problem here is to determine the variance associated with the set of estimates $\{v(t)\}$ for the three cases of $\{f_i(t)\}$ to be considered. In general, the variance is defined by

$$\sigma_v^2(t) = \langle v^2(t) \rangle - \left(\langle v(t) \rangle \right)^2 \quad (3.3)$$

From Eq. (3.1), the mean square ensemble average $\left\langle v^2(t)\right\rangle$, the first term on the right hand side of Eq. (3.3), is given by

$$\begin{aligned} \left\langle v^2(t)\right\rangle &= \frac{1}{N^2}\sum_{i,j=1}^{N}\left\langle f_i^2(t)\, f_j^2(t)\right\rangle \\ &= \frac{1}{N^2}\left[\sum_{i=1}^{N}\left\langle f_i^4(t)\right\rangle + \sum_{\substack{i,j=1\\ i\neq j}}^{N}\left\langle f_i^2(t)\, f_j^2(t)\right\rangle\right] \end{aligned} \tag{3.4}$$

Thus, the problem reduces to evaluation of the two ensemble averages

$$\left\langle f_i^4(t)\right\rangle \quad \text{and} \quad \left\langle f_i^2(t)\, f_j^2(t)\right\rangle \quad \text{where} \quad i \neq j$$

3.2 THREE MATHEMATICAL MODELS

Results will now be developed for three mathematical models of physical problems: Case 1. Independent Samples; Case 2. Dependent Samples; and, Case 3. Correlated Samples.

Case 1. Independent Samples

For this case, it is assumed that the records

$$\begin{aligned} f_i(t) &= s_i(t) + n_i(t) \\ f_j(t) &= s_j(t) + n_j(t) \end{aligned} \tag{3.5}$$

are such that $\left\langle n_i(t)\right\rangle = 0$ and

$$\begin{aligned} \left\langle f_i(t)\right\rangle &= \left\langle s_i(t)\right\rangle = \mu(t) \\ \left\langle f_i^2(t)\right\rangle &= \left\langle s_i^2(t)\right\rangle + \left\langle n_i^2(t)\right\rangle \end{aligned} \tag{3.6}$$

For $i \neq j$, the independence assumption requires

$$\left\langle f_i(t)\, f_j(t)\right\rangle = \left\langle s_i(t)\right\rangle \left\langle s_j(t)\right\rangle = \mu^2(t) \tag{3.7}$$

It follows that the variance

$$\sigma_f^2(t) = \left\langle f_i^2(t) \right\rangle - \mu^2(t) = \sigma_s^2(t) + \sigma_n^2(t) \tag{3.8}$$

where

$$\sigma_s^2(t) = \left\langle s_i^2(t) \right\rangle - \mu^2(t) \quad \text{and} \quad \sigma_n^2(t) = \left\langle n_i^2(t) \right\rangle \tag{3.9}$$

In order to obtain reasonable closed-form quantitative expressions in the following analysis, and also because it describes fairly closely a wide variety of situations, it will be assumed now that the set of values $\{f_i(t)\}$ at any time t follows a Gaussian (normal) distribution with mean value $\mu(t)$ and variance $\sigma_f^2(t)$. One can then derive for the higher moments

$$\left\langle f_i^4(t) \right\rangle = 3\left[\sigma_f^2(t) + \mu^2(t)\right]^2 - 2\,\mu^4(t) \tag{3.10}$$

while, for $i \neq j$, for Case 1,

$$\left\langle f_i^2(t)\, f_j^2(t) \right\rangle = \left[\sigma_f^2(t) + \mu^2(t)\right]^2 \tag{3.11}$$

Eqs. (3.10) and (3.11) follow from the Gaussian relation

$$\begin{aligned}\left\langle f_i(t)\, f_m(t)\, f_j(t)\, f_n(t) \right\rangle &= \left\langle f_i(t)\, f_j(t) \right\rangle \left\langle f_m(t)\, f_n(t) \right\rangle + \left\langle f_i(t)\, f_m(t) \right\rangle \left\langle f_j(t)\, f_n(t) \right\rangle \\ &\quad + \left\langle f_i(t)\, f_n(t) \right\rangle \left\langle f_j(t)\, f_m(t) \right\rangle - 2\,\mu^4(t)\end{aligned} \tag{3.12}$$

as applied to the two cases when $j = m = n = i$, and when $m = i$, $n = j$, $i \neq j$. In deriving **Eq. (3.11)**, use is made of the independence relation of **Eq. (3.7)**.

From Eqs. (3.8) and (3.10), the variance of the input squared values

$$\sigma_{f^2}^2(t) = \left\langle f_i^4(t) \right\rangle - \left[\left\langle f_i^2(t) \right\rangle\right]^2 = 2\left[\sigma_f^4(t) + 2\,\mu^2(t)\,\sigma_f^2(t)\right] \tag{3.13}$$

Substitution of Eqs. (3.10) and (3.11) into Eq. (3.4) yields

$$\langle v^2(t)\rangle = \frac{1}{N^2}\left\{N\left[3\left(\sigma_f^2(t) + \mu^2(t)\right)^2 - 2\mu^4(t)\right] + (N^2 - N)\left[\sigma_f^2(t) + \mu^2(t)\right]^2\right\}$$

$$= \left[\sigma_f^2(t) + \mu^2(t)\right]^2 + \frac{2}{N}\left\{\left[\sigma_f^2(t) + \mu^2(t)\right]^2 - \mu^4(t)\right\}$$

$$= \left(\langle v(t)\rangle\right)^2 + \frac{2}{N}\left[\sigma_f^4(t) + 2\sigma_f^2(t)\,\mu^2(t)\right] \tag{3.14}$$

Thus, the variance

$$\sigma_v^2(t) = \frac{2}{N}\left[\sigma_f^4(t) + 2\sigma_f^2(t)\,\mu^2(t)\right] \tag{3.15}$$

Note that $\sigma_v^2(t)$ approaches zero as N becomes large so the estimate $v(t)$ is a consistent estimate of $\mu_v(t)$.

Case 1. Signal-to-Noise Power Ratios

In terms of signal and noise variances, $\sigma_s^2(t)$ and $\sigma_n^2(t)$, Eqs. (3.2) and (3.14) become

$$\mu_v(t) = \langle v(t)\rangle = \sigma_s^2(t) + \mu^2(t) + \sigma_n^2(t) \tag{3.16}$$

$$\sigma_v^2(t) = \frac{2}{N}\left\{\left[\sigma_s^2(t) + \sigma_n^2(t)\right]^2 + 2\,\mu^2(t)\left[\sigma_s^2(t) + \sigma_n^2(t)\right]\right\} \tag{3.17}$$

An output signal-to-noise (power) ratio $P_o(t)$ will be defined by

$$P_o(t) = \frac{\mu_v^2(t)}{\sigma_v^2(t)} \tag{3.18}$$

It follows that

$$P_o(t) = N P_i(t) \tag{3.19}$$

where $P_i(t)$ is the input signal-to-noise (power) ratio defined by

$$P_i(t) = \frac{\left[\left\langle f_i^2(t)\right\rangle\right]^2}{\sigma^2_{f^2}(t)} = \frac{\left[\sigma_f^2(t) + \mu^2(t)\right]^2}{2\left[\sigma_f^4(t) + 2\,\mu^2(t)\sigma_f^2(t)\right]} \tag{3.20}$$

The input quantity $P_i(t) < 1$ whenever $\left[\mu(t)/\sigma_f(t)\right]^2 < \left|1 + \sqrt{2}\right|$. The output quantity $P_o(t)$ is directly proportional to the number N of available records.

Three special examples of Eqs. (3.15) through (3.20) will now be examined. It is assumed in all examples that $\sigma_s^2(t) \neq 0$.

Special Examples of Case 1:

(1) $\mu(t) = 0, \quad \sigma_n^2(t) \neq 0$

For this special example, results are

$$\mu_v(t) = \sigma_s^2(t) + \sigma_n^2(t)$$

$$\sigma_v^2(t) = \frac{2}{N}\left[\sigma_s^2(t) + \sigma_n^2(t)\right]^2$$

$$P_o(t) = \frac{N}{2} \quad ; \quad P_i(t) = \frac{1}{2} \tag{3.21}$$

Equation (3.21) shows that $P_o(t)$ is a function here only of N.

(2) $\mu(t) \neq 0, \quad \sigma_n^2(t) = 0$

For this situation

$$\mu_v(t) = \sigma_s^2(t) + \mu^2(t)$$

$$\sigma_v^2(t) = \frac{2}{N}\left[\sigma_s^4(t) + 2\,\mu^2(t)\sigma_s^2(t)\right]$$

$$P_o(t) = \frac{N\left[1 + 2\left(\mu(t)/\sigma_s(t)\right)^2 + \left(\mu(t)/\sigma_s(t)\right)^4\right]}{2\left[1 + 2\left(\mu(t)/\sigma_s(t)\right)^2\right]} \tag{3.22}$$

which shows that $P_o(t)$ here is a function of N and the dimensionless ratio $(\mu(t)/\sigma_s(t))$.

For small (μ/σ_s), $P_o(t) \approx (N/2)$

while for large (μ/σ_s), $P_o(t) \approx (N/4)(\mu/\sigma_s)^2$.

(3) $\mu(t) \neq 0$, $\sigma_n^2(t) \neq 0$

Upon dividing Eq. (3.17) by $\sigma_s^2(t)$, one obtains the general result

$$P_o(t) = \frac{N\left[1 + 2(\sigma_n/\sigma_s)^2 + (\sigma_n/\sigma_s)^4 + 2(\mu/\sigma_s)^2 + 2(\mu/\sigma_s)^2(\sigma_n/\sigma_s)^2 + (\mu/\sigma_s)^4\right]}{2\left[1 + 2(\sigma_n/\sigma_s)^2 + (\sigma_n/\sigma_s)^4 + 2(\mu/\sigma_s)^2 + 2(\mu/\sigma_s)^2(\sigma_n/\sigma_s)^2\right]} \tag{3.23}$$

using the simplified notation

$$(\sigma_n/\sigma_s) = \sigma_n(t)/\sigma_s(t)$$

$$(\mu/\sigma_s) = \mu(t)/\sigma_s(t)$$

Equation (3.23) shows that $P_o(t)$ is a function of N, (σ_n/σ_s) and (μ/σ_s). Note that Eq. (3.23) reduces to Eqs. (3.21) and (3.22) for those situations.

Various calculations can be made from Eq. (3.23) based on assumed values for N, (σ_n/σ_s), and (μ/σ_s). These are left as exercises when needed.

Example: Assume $(\mu/\sigma_s) = 1.0$

Then

$$P_o(t) = \frac{N\left[4 + 4(\sigma_n/\sigma_s)^2 + (\sigma_n/\sigma_s)^4\right]}{2\left[3 + 4(\sigma_n/\sigma_s)^2 + (\sigma_n/\sigma_s)^4\right]}$$

For $(\sigma_n/\sigma_s) = 0$, $P_o = \frac{2}{3}N$; For $(\sigma_n/\sigma_s) = 2$, $P_o = \frac{28}{54}N$

For $(\sigma_n/\sigma_s) = 1$, $P_o = \frac{9}{16}N$; For $(\sigma_n/\sigma_s) = 5$, $P_o = \frac{729}{1456}N$

Thus, for fixed (μ/σ_s) and fixed (σ_n/σ_s), P_o is a simple linear function of N as pictured in Figure 3.1.

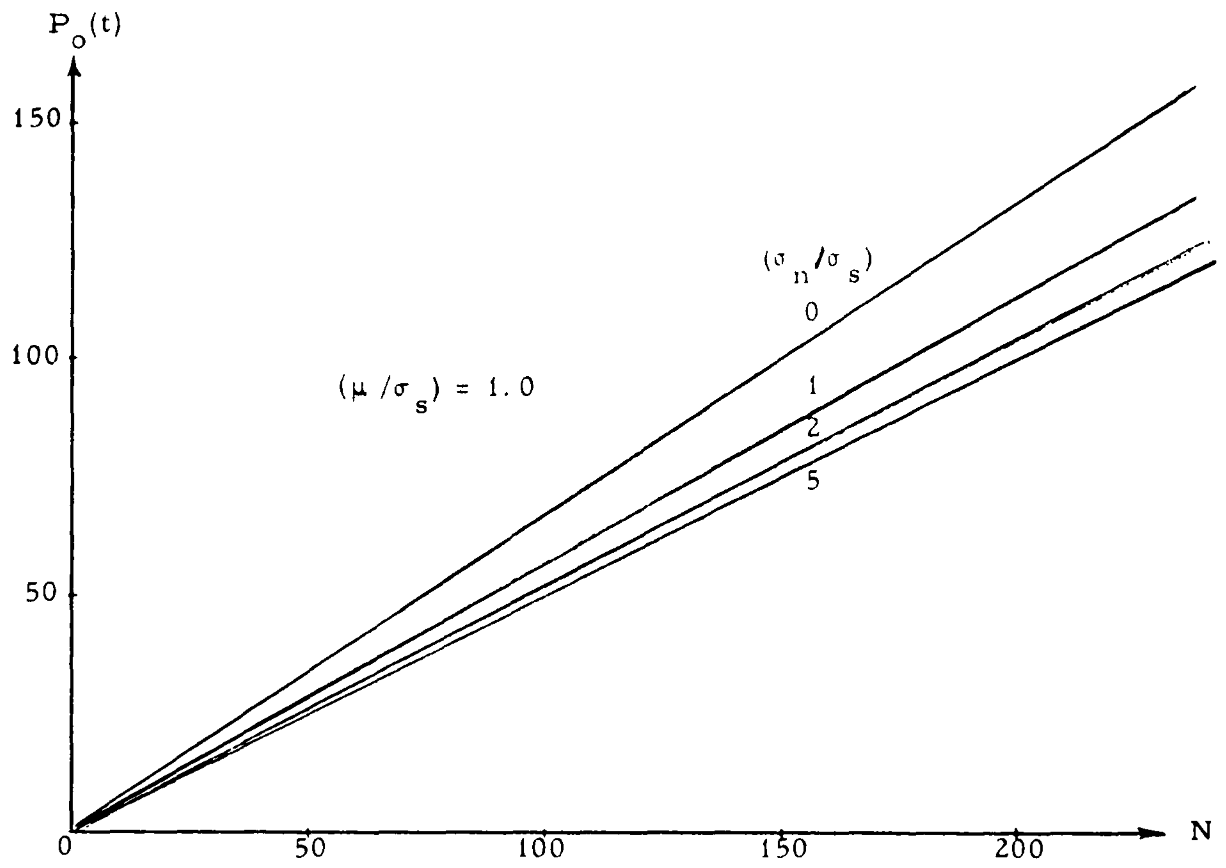

Figure 3.1. Plot of $P_o(t)$ as function of N for given (m/σ_s) and (σ_n/σ_s)

Case 1. Confidence Bands

From Eqs. (3.16) and (3.17) one can obtain confidence bands which will indicate how closely a measured mean square value will approximate the true mean square value at any time. This is accomplished as follows:

For an arbitrary distribution associated with the measurements $\{v(t)\}$, from the Tchebycheff Inequality, for any constant λ

$$\text{Prob}\left[|v(t) - \mu_v(t)| \geq \lambda\sigma_v(t) \right] \leq \frac{1}{\lambda^2} \tag{3.24}$$

where $\mu_v(t)$ and $\sigma_v(t)$ represent the true mean value and standard deviation associated with $\{v(t)\}$. Now, consider the range for estimating $\mu_v(t)$ from $v(t)$ as given by

$$|v(t) - \mu_v(t)| = \lambda\sigma_v(t) \tag{3.25}$$

Solving for v(t) yields the two extreme range values

$$v(t) = \mu_v(t) \pm \lambda\, \sigma_v(t) \tag{3.26}$$

Now for $\lambda = 3$, Eq. (3.24) shows that there is 89% confidence that v(t) lies in the range $\left[\mu_v(t) - 3\sigma_v(t), \mu_v(t) + 3\sigma_v(t)\right]$.

Equation (3.26) can be put into another equivalent form by using Eq.(3.18). This yields

$$v(t) = \mu_v(t) \pm \lambda \frac{\mu_v(t)}{\sqrt{P_o(t)}} \tag{3.27}$$

Hence

$$\mu_v(t) = \frac{v(t)}{1 \pm \left[\lambda/\sqrt{P_o(t)}\right]} \quad \text{provided} \quad \sqrt{P_o(t)} > \lambda \tag{3.28}$$

From Eq. (3.28), one can state at any desired confidence level how closely a measured mean square value v(t) will approximate the true mean square value $\mu_v(t)$.

For example, if $\lambda = 3$ and $P_o(t) = 25$, then there is 89% confidence that $\mu_v(t)$ lies in the range bounded by

$$\mu_v(t) = \frac{v(t)}{1 \pm 0.60} = \left[0.625\, v(t) \text{ to } 2.5\, v(t)\right]$$

For $P_o(t) = 100$ and $\lambda = 3$, the 89% confidence range becomes

$$\mu_v(t) = \frac{v(t)}{1 \pm 0.30} = \left[0.77\, v(t) \text{ to } 1.43\, v(t)\right]$$

Figure 3.2 based on Eq. (3. 28) is a plot of the ratio $\mu_v(t)/v(t)$ as a function of $P_o(t)$. Two cases are considered. Case 1 applies to arbitrary distributions for $v(t)$ and sets $\lambda = 3$ corresponding to an 89% confidence band as given by the Tchebycheff Inequality. Case 2 applies to a Gaussian probability distribution for $v(t)$ and sets $\lambda = 2$ corresponding to a 95% Gaussian confidence band. The lower and upper limits used for Figure 3. 2 are shown in Table 3. 1.

$P_o(t)$	Case 1		Case 2	
	Lower Limit	Upper Limit	Lower Limit	Upper Limit
4	---	---	0. 50	∞
9	0. 50	∞	0. 60	3. 00
16	0. 57	4. 00	0. 67	2. 00
25	0. 63	2. 50	0. 71	1. 67
36	0. 67	2. 00	0. 75	1. 50
64	0. 73	1. 60	0. 80	1. 33
100	0. 77	1. 43	0. 83	1. 25
144	0. 80	1. 33	0. 86	1. 20
225	0. 83	1. 25	0. 88	1. 15
324	0. 86	1. 20	0. 90	1. 12
400	0. 87	1. 18	0. 91	1. 11

Table 3. 1 Data for Figure 3. 2

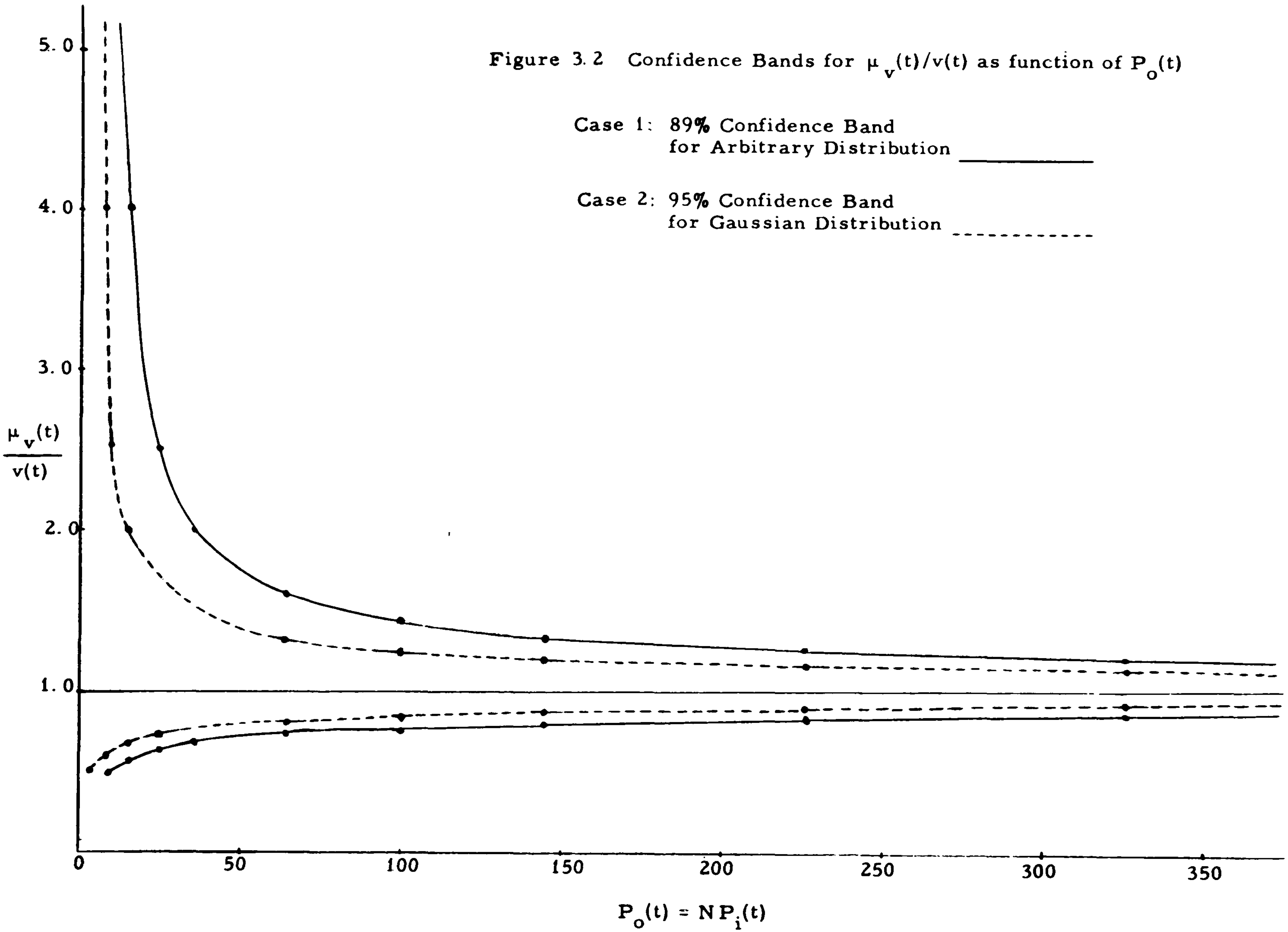

Figure 3.2 Confidence Bands for $\mu_v(t)/v(t)$ as function of $P_o(t)$

Case 2. Dependent Samples

This case differs from Case 1 in that Eq. (3.7) where $i \neq j$ is replaced by

$$\left\langle f_i(t) f_j(t) \right\rangle = \left\langle s_i^2(t) \right\rangle = \sigma_s^2(t) + \mu^2(t) \tag{3.29}$$

It follows that Eq. (3.11) where $i \neq j$ is then replaced by

$$\left\langle f_i^2(t) f_j^2(t) \right\rangle = \left[\sigma_f^2(t) + \mu^2(t)\right]^2 + 2\left[\sigma_s^4(t) + 2\mu^2(t)\sigma_s^2(t)\right] \tag{3.30}$$

Substitution of Eqs. (3.10) and (3.30) into Eq. (3.4) now yields all that was obtained before in Eq. (3.14) plus an additional term, namely,

$$\left\langle v^2(t) \right\rangle = \left(\left\langle v(t) \right\rangle\right)^2 + \frac{2}{N}\left[\sigma_f^4(t) + 2\mu^2(t)\sigma_f^2(t)\right] + 2\frac{(N^2 - N)}{N^2}\left[\sigma_s^4(t) + 2\mu^2(t)\sigma_s^2(t)\right] \tag{3.31}$$

It then follows from Eqs. (3.3) and (3.8), that the variance

$$\sigma_v^2(t) = 2\sigma_s^2(t)\left[\sigma_s^2(t) + 2\mu^2(t)\right] + \frac{2}{N}\sigma_n^2(t)\left[\sigma_n^2(t) + 2\mu^2(t)\right] \tag{3.32}$$

Equation (3.32) should be compared with Eq. (3.17) from Case 1. Note that if $\sigma_s^2(t) \neq 0$, then $\sigma_v^2(t)$ no longer approaches zero as N becomes large so that in Case 2 the estimate $v(t)$ is not a consistent estimate.

As in Case 1, an output signal-to-noise (power) ratio $P_o(t)$ will be defined by

$$P_o(t) = \frac{\mu_v^2(t)}{\sigma_v^2(t)} \tag{3.33}$$

For Case 2, $\mu_v(t)$ is given by Eq. (3.16) and $\sigma_v^2(t)$ by Eq. (3.32).

Two special examples of Eqs. (3.32) and (3.33) will be examined.

Special Examples of Case 2:

(1) $\mu(t) = 0, \quad \sigma_n^2(t) \neq 0$

For this example $\mu_v(t) = \sigma_s^2(t) + \sigma_n^2(t)$, and

$$\sigma_v^2(t) = 2\sigma_s^4(t) + \frac{2}{N}\sigma_n^4(t) = \frac{2}{N}\left[N\sigma_s^4(t) + \sigma_n^4(t)\right]$$

Hence

$$P_o(t) = \frac{N\left[1 + 2(\sigma_n/\sigma_s)^2 + (\sigma_n/\sigma_s)^4\right]}{2\left[N + (\sigma_n/\sigma_s)^4\right]} \qquad (3.34)$$

Thus, $P_o(t)$ is a function here of both N and (σ_n/σ_s).

For small (σ_n/σ_s), $P_o(t) \approx 0.50$

while for large (σ_n/σ_s), $P_o(t) \approx (N/2)$.

(2) $\mu(t) \neq 0, \quad \sigma_n^2(t) = 0$

For this example, $\mu_v(t) = \sigma_s^2(t) + \mu^2(t)$, and

$$\sigma_v^2(t) = 2\sigma_s^2(t)\left[\sigma_s^2(t) + 2\mu^2(t)\right]$$

Hence

$$P_o(t) = \frac{1 + 2(\mu/\sigma_s)^2 + (\mu/\sigma_s)^4}{2 + 4(\mu/\sigma_s)^2} \qquad (3.35)$$

For small (μ/σ_s), $P_o(t) \approx 0.50$

while for large (μ/σ_s), $P_o(t) \approx 0.25(\mu/\sigma_s)^2$

Case 3. Correlated Samples

For this case, it will be assumed that

$$\left\langle f_i(t)\, f_j(t) \right\rangle = \left\langle s_i(t)\, s_j(t) \right\rangle = R_{ss}(k, t) \quad \text{where} \quad k = j - i \tag{3.36}$$

The quantity $R_{ss}(k, t)$ was defined previously in Eq. (2.42) dealing with the measurement of average response values for correlated data. Equation (3.11) where $i \neq j$ is now replaced by

$$\left\langle f_i^2(t)\, f_j^2(t) \right\rangle = \left[\sigma_f^2(t) + \mu^2(t)\right]^2 + 2\left[R_{ss}^2(k, t) - \mu^4(t)\right] \tag{3.37}$$

When $i = j$, Eq. (3.36) reduces to

$$\left\langle f_i^2(t) \right\rangle = \left\langle s_i^2(t) \right\rangle = R_{ss}(0, t) = \sigma_s^2(t) + \mu^2(t) \tag{3.38}$$

From Eqs. (3.37) and (3.38), Case 1 and Case 2 are thus seen to be special cases of Case 3, namely,

(1) $R_{ss}(k, t) = \mu^2(t)$ for all $k > 1$ yields Case 1

(2) $R_{ss}(k, t) = R_{ss}(0, t)$ for all $k > 1$ yields Case 2

A procedure for handling $R_{ss}(k, t)$ was developed in the previous material on average response computers. A similar analysis carried out here yields the result

$$\sigma_v^2(t) = \frac{2}{N}\left[\sigma_f^4(t) + \sigma_f^2(t)\,\mu^2(t)\right] + \frac{4}{N^2}\sum_{k=1}^{N-1}(N-k)\left[R_{ss}^2(k, t) - \mu^4(t)\right] \tag{3.39}$$

Equation (3.39) is a major result of this investigation. It includes Eqs. (3.17) and (3.32) as special cases. As discussed previously $R_{ss}(k, t)$ can be interpreted as a measure of adaptive changes in records occurring from repeated stimuli. Thus, $R_{ss}(k, t)$ may be useful for analyzing such items as structural fatigue effects, or learning patterns in physiological systems.

Exponential Correlation Function

In order to obtain some quantitative expressions corresponding to Eq.(3.39) which will characterize different degrees of correlation, assume that $R_s(k,t)$ has the exponential form

$$R_s(k,t) = \mu^2(t) + \sigma_s^2(t)\, e^{-|k|c} \qquad (3.40)$$

where k and c are positive constants. The parameter c, called the correlation coefficient, is such that a large value of c corresponds to Case 1 while a small value for c yields Case 2. Thus the value of c may be used to compare different physical situations.

The sum in Eq.(3.39) may be evaluated as follows:

Let

$$f(c) = \sum_{k=1}^{N-1} e^{-kc} = \frac{1 - e^{-(N-1)c}}{e^c - 1} \qquad (3.41)$$

Then

$$F(c) = \sum_{k=1}^{N-1} (N-k)\, e^{-kc} = N f(c) + f'(c) = \frac{(N-1)e^c - N + e^{-(N-1)c}}{(e^c - 1)^2} \qquad (3.42)$$

From Eq. (3.40), the term

$$R_{ss}^2(k,t) - \mu^4(t) = \sigma_s^2(t)\left[2\mu^2(t)\, e^{-kc} + \sigma_s^2(t)\, e^{-2kc}\right] \qquad (3.43)$$

Hence

$$\frac{4}{N^2} \sum_{k=1}^{N-1} (N-k)\left[R_{ss}^2(k,t) - \mu^4(t)\right] = \frac{4\sigma_s^2(t)}{N^2} \sum_{k=1}^{N-1} (N-k)\left[2\mu^2(t)\, e^{-kc} + \sigma^2(t)\, e^{-2kc}\right]$$

$$= \frac{4\sigma_s^2(t)}{N^2}\left[2\mu^2(t)\, F(c) + \sigma_s^2(t)\, F(2c)\right] \qquad (3.44)$$

where F(c) is given by Eq. (3.42).

Equation (3.44) can be used to generate a set of curves for different values of N and c. For a given value of N, and for known values of $\mu(t)$ and $\sigma_s(t)$, actual experimental results would enable one to estimate the correlation coefficient c.

3.3 CONCLUSION

This material on mean square value measurements, together with the previous material in Section 2 on average response (mean value) measurements, provides basic formulas for analyzing a wide class of nonstationary data occurring in physical problems. For given input conditions, one can decide in advance how large the sample size (number of records) should be to guarantee a desired confidence in final results. Conversely, for a given sample size and for given input conditions, one can determine confidence bands about measured results.

Various special techniques are developed here which may prove useful also for other problems. In particular, the mathematical model which has been considered for correlated samples offers a new procedure for analyzing fatigue effects or other adaptive physical changes.

4. SPECTRAL DECOMPOSITION OF NONSTATIONARY PROCESSES

4.1 POWER SPECTRAL DENSITY FUNCTIONS FOR STATIONARY PROCESSES

For a single stationary record $x(t)$, the power spectral density function defines the rate of change of the mean square value of the record with frequency. The mean square value inside any frequency band is found by merely summing the power spectral density function across the band. In equation form, if $S_x(f)$ is the stationary two-sided power spectral density function of $x(t)$, defined over $(-\infty, \infty)$, then the mean square values $\overline{x^2(f_1, f_2)}$ inside (f_1, f_2) is given by

$$\overline{x^2(f_1, f_2)} = 2\int_{f_1}^{f_2} S_x(f)\, df \qquad (4.1)$$

where the factor 2 is due to the two-sided nature of $S_x(f)$.

The same formula describes the average power inside (f_1, f_2) for a stationary random process $\{x(t)\}$ by interpreting $S_x(f)$ as the ensemble average of all the individual stationary power spectral density functions in the random process.

One should note that the result of Eq. (4.1) is independent of t since the random process is assumed to be stationary. For a stationary process, one can prove that $S_x(f)$ is the Fourier transform of the stationary autocorrelation function $R_x(\tau)$, namely,

$$S_x(f) = \int_{-\infty}^{\infty} R_x(\tau)\, e^{-j2\pi f\tau}\, d\tau \qquad (4.2)$$

where $R_x(\tau)$ is defined by the ensemble average

$$R_x(\tau) = \langle x(t)\, x(t+\tau)\rangle \quad , \quad \text{independent of } t \qquad (4.3)$$

In terms of $S_x(f)$, the inverse Fourier transform to Eq. (4. 2) yields

$$R_x(\tau) = \int_{-\infty}^{\infty} S_x(f)\, e^{j2\pi f\tau}\, df \tag{4. 4}$$

The total mean square value is independent of t and is given by

$$R_x(0) = \left\langle x^2(t) \right\rangle = \int_{-\infty}^{\infty} S_x(f)\, df = 2\int_0^{\infty} S_x(f)\, df \tag{4. 5}$$

When a random process is nonstationary, the nonstationary autocorrelation function is no longer a function of a single time variable but instead is a function of two time variables. The corresponding nonstationary power spectral density is also no longer a function of a single frequency variable but instead is a function of two frequency variables. By proper definitions and interpretation, one can show that nonstationary autocorrelation functions and corresponding nonstationary power spectral density functions are double Fourier transforms of one another, which reduce to the above formulas in the stationary case. A similar correspondance exists for nonstationary cross-power spectral density functions and nonstationary cross-correlation functions. This procedure will now be carried out, and appropriate expressions will be derived which describe general nonstationary processes.

A separate section will consider special situations of nonstationary random processes which are locally stationary. These represent important physical cases, such as turbulence, where the average instantaneous power may vary slowly with respect to the correlation time.

4.2 CORRELATION (COVARIANCE) STRUCTURE OF NONSTATIONARY PROCESSES

For general real-valued nonstationary random processes, $\left\{{}^{k}x(t)\right\}$, $\left\{{}^{k}y(t)\right\}$, the mean values at arbitrary fixed values of time t are defined by the ensemble averages

$$\mu_x(t) = \left\langle {}^{k}x(t) \right\rangle_{\text{Av over } k} = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} {}^{k}x(t) \qquad ; \quad t \text{ fixed}$$

$$\mu_y(t) = \left\langle {}^{k}y(t) \right\rangle_{\text{Av over } k} = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} {}^{k}y(t) \qquad ; \quad t \text{ fixed} \tag{4.6}$$

The record index k is averaged out in computing these ensemble averages which are indicated by angular brackets. For simplicity in notation, the index k will be omitted in future equations. However, t must be retained since, in general, these mean values will change with time.

The covariance functions at arbitrary fixed values of t_1 and t_2 are defined by the ensemble averages

$$\rho_x(t_1, t_2) = \left\langle \left[x(t_1) - \mu_x(t_1)\right]\left[x(t_2) - \mu_x(t_2)\right]\right\rangle$$

$$\rho_y(t_1, t_2) = \left\langle \left[y(t_1) - \mu_y(t_1)\right]\left[y(t_2) - \mu_y(t_2)\right]\right\rangle \tag{4.7}$$

$$\rho_{xy}(t_1, t_2) = \left\langle \left[x(t_1) - \mu_x(t_1)\right]\left[y(t_2) - \mu_y(t_2)\right]\right\rangle$$

For stationary random processes, these results would be a function of the difference $(t_1 - t_2)$ or $(t_2 - t_1)$ rather than t_1 or t_2. In general, however, these quantities will vary with both t_1 and t_2. At the special point $t = t_1 = t_2$, the covariance functions $\rho_x(t, t)$ and $\rho_y(t, t)$ represent the ordinary variances of $\left\{x(t)\right\}$ and $\left\{y(t)\right\}$ at a fixed value of t, while $\rho_{xy}(t, t)$ represents the cross-covariance between $\left\{x(t)\right\}$ and $\left\{y(t)\right\}$.

The <u>correlation functions</u> for nonstationary processes are defined by

$$R_x(t_1, t_2) = \langle x(t_1)\, x(t_2) \rangle$$
$$R_y(t_1, t_2) = \langle y(t_1)\, y(t_2) \rangle \tag{4.8}$$
$$R_{xy}(t_1, t_2) = \langle x(t_1)\, y(t_2) \rangle$$

where R is introduced instead of ρ to agree with engineering usage. Note that for zero mean values, the correlation functions are the same as the covariance functions. The quantities $R_x(t_1, t_2)$ and $R_y(t_1, t_2)$ are nonstationary <u>autocorrelation</u> functions, while $R_{xy}(t_1, t_2)$ is a nonstationary <u>cross-correlation</u> function. The cross-correlation function R_{xy} includes the autocorrelation function as a special case when $x = y$.

For arbitrary values of $\mu_x(t)$ and $\mu_y(t)$ the covariance functions and correlation functions are related by

$$\rho_x(t_1, t_2) = R_x(t_1, t_2) - \mu_x(t_1)\mu_x(t_2)$$
$$\rho_y(t_1, t_2) = R_y(t_1, t_2) - \mu_y(t_1)\mu_y(t_2) \tag{4.9}$$
$$\rho_{xy}(t_1, t_2) = R_{xy}(t_1, t_2) - \mu_x(t_1)\mu_y(t_2)$$

An upper bound for the cross-correlation (or autocorrelation) function is given by the inequality

$$|R_{xy}(t_1, t_2)|^2 \leqq R_x(t_1, t_1)\, R_y(t_2, t_2) \tag{4.10}$$

From the original definitions, one sees that the following symmetry properties are satisfied.

$$R_x(t_2, t_1) = R_x(t_1, t_2)$$
$$R_y(t_2, t_1) = R_y(t_1, t_2) \tag{4.11}$$
$$R_{xy}(t_2, t_1) = R_{yx}(t_1, t_2)$$

The correlation structure of nonstationary random processes $\{x(t)\}$, $\{y(t)\}$ may be described by the four functions $R_x(t_1, t_2)$, $R_y(t_1, t_2)$, $R_{xy}(t_1, t_2)$ and $R_{yx}(t_1, t_2)$. These need be calculated only for values of $t_1 \leqq t_2$ since the symmetry properties of Eq (4.11) yield results for $t_2 \leqq t_1$. When the mean values are zero, for arbitrary t, $R_x(t, t)$ and $R_y(t, t)$ represent the variance associated with $\{x(t)\}$ and $\{y(t)\}$, respectively, while $R_{xy}(t, t)$ represents the cross-covariance between $\{x(t)\}$ and $\{y(t)\}$.

Consider the problem of measuring $R_x(t_1, t_2)$. By definition

$$R_x(t_1, t_2) = \langle x(t_1)\, x(t_2) \rangle$$

Theoretically, all possible combinations of the two times t_1 and t_2 should be used in computing $R_x(t_1, t_2)$, a tremendous task. To lighten this work, a special cross-section can be studied.

A recommended procedure is as follows. Hold t_1 fixed and vary t_2. Let $t_2 = t_1 + \tau$ with $\tau \geqq 0$. This yields

$$R_x(t_1, t_1 + \tau) = \langle x(t_1)\, x(t_1 + \tau) \rangle$$

For fixed t_1, this result is a function only of τ. By varying t_1 in small steps, the dependence on t_1 will be exhibited. For stationary random processes, there would be no dependence on t_1. Clearly, if $R_x(t_1, t_1 + \tau)$ is determined for all possible t_1 and all possible $\tau \geqq 0$, then one should be able to state the general form of $R_x(t_1, t_2)$.

<u>Example</u>: Suppose $R_x(t_1, t_1 + \tau) = \cos \Omega\tau + \cos \Omega(2t_1 + \tau)$. By letting $\tau = t_2 - t_1$, one finds from a trigonometric identity,

$$R_x(t_1, t_2) = 2 \cos \Omega t_1 \cos \Omega t_2$$

4.3 SPECTRAL DECOMPOSITION OF NONSTATIONARY PROCESSES

To obtain a spectral decomposition for a pair of random processes $\{x(t)\}$ and $\{y(t)\}$, it will now be assumed that each particular member of either random processes has a complex Fourier transform

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\, dt$$
$$Y(f) = \int_{-\infty}^{\infty} y(t)\, e^{-j2\pi ft}\, dt \tag{4.12}$$

The parameter f denotes the frequency (usually cps) while $j = \sqrt{-1}$.
Inverse Fourier transforms yield

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi ft}\, df$$
$$y(t) = \int_{-\infty}^{\infty} Y(f)\, e^{j2\pi ft}\, df \tag{4.13}$$

Thus, an original pair of real-valued nonstationary random processes $\{x(t)\}$, $\{y(t)\}$ may be described in terms of two new complex-valued random processes $\{X(f)\}$, $\{Y(f)\}$.

When $x(t)$ is real-valued, it follows that the complex conjugate $\overline{X(f)}$ is given by

$$\overline{X(f)} = \int_{-\infty}^{\infty} x(t)\, e^{j2\pi ft}\, dt = X(-f) \tag{4.14}$$

Also, since $x(t)$ is real-valued,

$$x(t) = \int_{-\infty}^{\infty} \overline{X(f)}\, e^{-j2\pi ft}\, df \tag{4.15}$$

Consider now the nonstationary correlation function $R_x(t_1, t_2)$. From Eqs.(4. 13) and (4. 15), one finds

$$x(t_1)\,x(t_2) \doteq \int_{-\infty}^{\infty} \overline{X(f_1)}\, e^{-j2\pi f_1 t_1}\, df_1 \int_{-\infty}^{\infty} X(f_2)\, e^{j2\pi f_2 t_2}\, df_2$$

using a variable f_2 instead of f_1 in the second integral to avoid confusion. The ensemble average yields

$$R_x(t_1, t_2) = \left\langle x(t_1)\,x(t_2)\right\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_x(f_1, f_2)\, e^{-j2\pi(f_1 t_1 - f_2 t_2)}\, df_1\, df_2 \tag{4. 16}$$

where

$$S_x(f_1, f_2) = \left\langle \overline{X(f_1)}\, X(f_2)\right\rangle \tag{4. 17}$$

By inversion,

$$S_x(f_1, f_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_x(t_1, t_2)\, e^{j2\pi(f_1 t_1 - f_2 t_2)}\, dt_1\, dt_2 \tag{4. 18}$$

The quantity $S_x(f_1, f_2)$ is called the generalized (nonstationary) power spectral density function of the random process $\left\{x(t)\right\}$.

For general cases, from Eq. (4. 16),the variance

$$R_x(t, t) = \left\langle x^2(t)\right\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_x(f_1, f_2)\, e^{j2\pi t(f_2 - f_1)}\, df_1\, df_2 \tag{4. 19}$$

Similarly, one derives for the nonstationary cross-correlation function

$$R_{xy}(t_1, t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_{xy}(f_1, f_2)\, e^{-j2\pi(f_1 t_1 - f_2 t_2)}\, df_1\, df_2$$

and (4. 20)

$$S_{xy}(f_1, f_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{xy}(t_1, t_2)\, e^{j2\pi(f_1 t_1 - f_2 t_2)}\, dt_1\, dt_2$$

where

$$S_{xy}(f_1, f_2) = \left\langle \overline{X(f_1)}\, Y(f_2) \right\rangle \tag{4.21}$$

The quantity $S_{xy}(f_1, f_2)$ is called the generalized (nonstationary) cross-power spectral density function between the pair of processes $\{x(t)\}$ and $\{y(t)\}$. One should note the basic symmetry between $R_{xy}(t_1, t_2)$ and $S_{xy}(f_1, f_2)$ as exhibited by Eq.(4.20).

From Eqs. (4.14), (4.17), and (4.21),

$$\begin{aligned} \overline{S_x(f_1, f_2)} &= S_x(f_2, f_1) = S_x(-f_1, -f_2) \\ \overline{S_{xy}(f_1, f_2)} &= S_{yx}(f_2, f_1) = S_{xy}(-f_1, -f_2) \end{aligned} \tag{4.22}$$

For the special point $t = t_1 = t_2$, Eq.(4.20) becomes

$$R_{xy}(t, t) = \left\langle x(t)\, y(t) \right\rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S_{xy}(f_1, f_2)\, e^{j2\pi t(f_2 - f_1)}\, df_1\, df_2 \tag{4.23}$$

Note that, for nonstationary random processes, the dependence on t is shown in the exponent in Eq.(4.23).

4.3.1 Stationary Random Processes

For stationary random processes, Eq. (4.23) would be independent of t, namely,

$$R_{xy}(t, t) = R_{xy}(0, 0) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S_{xy}(f_1, f_2)\, df_1\, df_2 \tag{4.24}$$

It follows that, for stationary random processes,

$$R_{xy}(0, 0) = \int_{-\infty}^{\infty} S_{xy}(f_2)\, df_2 \tag{4.25}$$

where

$$S_{xy}(f_2) = \int_{-\infty}^{\infty} S_{xy}(f_1, f_2)\, df_1 \qquad (4.26)$$

Hence, for stationary random processes,

$$S_{xy}(f_1, f_2) = S_{xy}(f_1)\delta(f_2 - f_1) \qquad (4.27)$$

where $\delta(f_2 - f_1)$ is the usual Dirac delta function at $f_2 = f_1$. In geometrical terms, $S_{xy}(f_1, f_2)$ is concentrated only on the line $f_2 = f_1$ in the (f_1, f_2) plane. Note also that $S_{xy}(f_1, f_2) = 0$ for $f_2 \neq f_1$. The delta function is an even function, so that one can replace $\delta(f)$ by $\delta(-f)$ whenever desired.

Conversely, suppose that a generalized cross-power spectral density function has the form

$$S_{xy}(f_1, f_2) = S_{xy}(f_1)\, \delta(f_2 - f_1) \qquad (4.28)$$

One can derive from Eq.(4.20), letting $\tau = t_2 - t_1$,

$$R_{xy}(t_1, t_2) = R_{xy}(t_2 - t_1) = R_{xy}(\tau) = \int_{-\infty}^{\infty} S_{xy}(f)\, e^{j2\pi f\tau}\, df \qquad (4.29)$$

where

$$S_{xy}(f) = \int_{-\infty}^{\infty} R_{xy}(\tau)\, e^{-j2\pi f\tau}\, d\tau \qquad (4.30)$$

Thus the cross-correlation function $R_{xy}(t_1, t_2)$ corresponding to Eq.(4.28), is stationary.

4.3.2 Nonstationary Random Processes

Return now to the consideration of general nonstationary processes. Through a change in variables, let

$$\tau = t_2 - t_1 \quad ; \quad t = \frac{t_1 + t_2}{2} \tag{4.31}$$

Then

$$t_1 = t - \frac{\tau}{2} \quad ; \quad t_2 = t + \frac{\tau}{2} \tag{4.32}$$

Thus, through the above change,

$$R_{xy}(t_1, t_2) = R_{xy}(t - \frac{\tau}{2}, t + \frac{\tau}{2}) = \mathcal{R}_{xy}(t, \tau) \tag{4.33}$$

The reason for introducing this transformation from the (t_1, t_2) plane to the (t, τ) plane is to separate, if possible, the nonstationary portion of the process from the stationary portion of the process. For stationary processes, there would be no dependence upon t. Note that $R_{xy}(t, t) = \mathcal{R}_{xy}(t, 0)$.

From Eqs. (4.11) and (4.33) one obtains

$$\begin{aligned} \mathcal{R}_x(t, -\tau) &= \mathcal{R}_x(t, \tau) \\ \mathcal{R}_{xy}(t, -\tau) &= \mathcal{R}_{yx}(t, \tau) \end{aligned} \tag{4.34}$$

Substitution of Eq. (4.32) into Eqs. (4.19) or (4.20) yields (omitting the subscripts x or xy)

$$S(f_1, f_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathcal{R}(t, \tau) \exp\left\{-j2\pi\left[(f_2 - f_1)t + \left(\frac{f_1 + f_2}{2}\right)\tau\right]\right\} dt\, d\tau$$

$$= \mathcal{S}\left[(f_2 - f_1), \left(\frac{f_1 + f_2}{2}\right)\right] = \mathcal{S}(f, g) = S(g - \frac{f}{2}, g + \frac{f}{2}) \tag{4.35}$$

through a change in variables where

$$f = f_2 - f_1 \quad ; \quad g = \frac{f_1 + f_2}{2}$$
$$f_1 = g - \frac{f}{2} \quad ; \quad f_2 = g + \frac{f}{2} \tag{4.36}$$

Thus $S(f_1, f_2)$ in the $(f_1 f_2)$ plane is transformed to $\mathscr{S}(f, g)$ in the (f, g) plane, and

$$\mathscr{S}(f, g) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathscr{R}(t, \tau)\, e^{-j2\pi(ft + g\tau)}\, gt\, d\tau \tag{4.37}$$

By inversion,

$$\mathscr{R}(t, \tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathscr{S}(f, g)\, e^{j2\pi(ft + g\tau)}\, df\, dg \tag{4.38}$$

This transformation to $\mathscr{S}(f, g)$ is required in order to bring out intrinsic frequency properties of the transformed nonstationary correlation function $\mathscr{R}(t, \tau)$. It will be shown shortly that for a stationary random process $\mathscr{S}(f, g)$ must be zero everywhere in the (f, g) plane except along the line $f = 0$.

Corresponding to Eq. (4.19), the variance

$$\mathscr{R}_x(t, 0) = \langle x^2(t) \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathscr{S}_x(f, g)\, e^{j2\pi ft}\, df\, dg$$

Note also that

$$\mathscr{R}_x(0, \tau) = \langle x(-\tau/2)\, x(\tau/2) \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathscr{S}_x(f, g)\, e^{j2\pi g\tau}\, df\, dg$$

4.3.3 Separable Case

Assume that the nonstationary correlation function $\mathcal{R}(t, \tau)$ can be separated into the products

$$\mathcal{R}(t, \tau) = R_1(t)\, R_2(\tau) = R_1\left(\frac{t_1 + t_2}{2}\right) R_2(t_2 - t_1) \qquad (4.39)$$

where $R_2(\tau) = R_2(t_2 - t_1)$ is a stationary correlation function, and $R_1(t) = R_1\left[(t_1 + t_2)/2\right]$ is a variable scale factor defined at the midpoint (average) of the points t_1 and t_2. For stationary random processes, $R_1(t)$ is a constant.

The process is said to be <u>locally stationary</u>, see Ref. [2], if $R_1(t)$ is a non-negative function of t. Although, in general, $R_1(t)$ is an arbitrary function of t, it may be possible to break up a random process into a number of smaller samples such that these smaller samples are stationary or locally stationary. For example, a physical process (such as turbulence phenomena) is locally stationary if its average instantaneous power is varying slowly with respect to its memory (correlation) time.

Substitution of Eq. (4.39) into Eq. (4.37) yields

$$\mathcal{S}(f, g) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_1(t)\, R_2(\tau)\, e^{-j2\pi(ft + g\tau)}\, dt\, d\tau$$

$$= S_1(f)\, S_2(g) \qquad (4.40)$$

where

$$S_1(f) = \int_{-\infty}^{\infty} R_1(t)\, e^{-j2\pi ft}\, dt \quad ; \quad f = f_2 - f_1$$

$$S_2(g) = \int_{-\infty}^{\infty} R_2(\tau)\, e^{-j2\pi g\tau}\, d\tau \quad ; \quad g = \frac{f_1 + f_2}{2} \qquad (4.41)$$

By inversion

$$R_1(t) = \int_{-\infty}^{\infty} S_1(f)\, e^{j2\pi ft}\, df$$
$$R_2(\tau) = \int_{-\infty}^{\infty} S_2(g)\, e^{j2\pi g\tau}\, dg \qquad (4.42)$$

For a stationary random process where $R_1(t)$ is a constant, say $R_1(t) = c$, the corresponding $S_1(f)$ is a delta function given by $S_1(f) = c\delta(f)$. Thus, for a stationary process,

$$\mathcal{R}(t, \tau) = c\, R_2(\tau)$$

with

$$\mathcal{S}(f, g) = c\, \delta(f)\, S_2(g) \qquad (4.43)$$

In words, when $R(t, \tau)$ has no dependence on t and is a function only of the time difference $\tau = t_2 - t_1$, the spectral density function $\mathcal{S}(f, g)$ is concentrated on the line $f = 0$ in the (f, g) plane.

For the special case of stationary "white" noise where $R_2(\tau) = \delta(\tau)$, a delta function, the corresponding $S_2(g) = 1$, a constant. Now,

$$\mathcal{R}(t, \tau) = c\, \delta(\tau)$$

with

$$\mathcal{S}(f, g) = c\delta(f) \qquad (4.44)$$

Thus, for this case, and this case alone, the spectral density function $\mathcal{S}(f, g)$ is itself stationary since there is dependence only on the frequency difference $f = f_2 - f_1$. Equation (4.43) shows that for general stationary random processes, the corresponding spectral density function is not stationary since there is dependence on $g = (f_1 + f_2)/2$.

4.3.4 Slowly Varying Nonstationary Random Processes

Suppose $\mathcal{R}(t, \tau)$ as defined by Eq. (4.33) varies slowly with respect to t. Then a single record, or a pair of records, can be broken up into a number of shorter records such that $\mathcal{R}(t, \tau)$ is essentially independent of t over the shorter records. Define

$$\mathcal{S}(g, t) = \int_{-\infty}^{\infty} \mathcal{R}(t, \tau) e^{-j2\pi g \tau} d\tau \quad (4.45)$$

When $\mathcal{R}(t, \tau)$ is independent of t, then $\mathcal{S}(g, t)$ is the power spectral density function of a stationary random process whose accuracy can be estimated by well known procedures.

The nonstationary generalized spectral density function $\mathcal{S}(f, g)$ can now be determined from $\mathcal{S}(g, t)$ by taking its complex Fourier transform with respect to t. This follows from Eq. (4.37) by noting that

$$\mathcal{S}(f, g) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \mathcal{R}(t, \tau) e^{-j2\pi g \tau} d\tau \right] e^{-j2\pi f t} dt$$

$$= \int_{-\infty}^{\infty} \mathcal{S}(g, t) e^{-j2\pi f t} dt \quad (4.46)$$

Further discussion of this idea and related matters concerning the analysis of nonstationary processes appears in Ref. [1].

4.4 REFERENCES

1 McLead, M. G., "Some Aspects of the Harmonic Analysis of Non-stationary Random Processes," MS Thesis, UCLA Dept. of Engineering, June 1959.

2 Silverman, R. A., "Locally Stationary Random Processes," IRE Trans. Inf. Theory, vol. IT-3, Sept. 1957, pp. 182-187.

5. INPUT-OUTPUT RELATIONS FOR NONSTATIONARY PROCESSES

5.1 INTRODUCTION

The subjects covered in this section are of considerable theoretical and practical importance for many problems of predicting output effects of nonstationary random processes from knowledge of input properties and system characteristics. For other problems, the input conditions, or the system behavior, can be predicted from measurement of the output properties. Optimum systems can also be derived from certain input-output information [1].

In the present discussion, emphasis will be placed on time-varying linear systems, and methods will be developed for describing such systems both in the time domain and the frequency domain. Physical situations of constant parameter linear systems are treated as a special subclass of time-varying linear systems.

After reviewing some of the basic properties of time-varying linear systems, linear, integral, and derivative transformations of random functions are considered in a general way. This is then followed by the main material in this section dealing with input-output relations of nonstationary random processes through time-varying linear systems.

5.2 TIME-VARYING LINEAR SYSTEMS

A time-varying linear system is characterized by a weighting function $h(\alpha, t)$ which, by definition, represents the response of the system at time t to a unit impulse function applied at time $t - \alpha$, that is α time units earlier. If the system response is independent of the particular time t at which the system is being observed and is a function only of α, namely $h(\alpha, t) = h(\alpha)$, then the system is said to be a constant parameter linear system. Thus, a constant parameter linear system is a special case of a time-varying linear system.

In order for a time-varying linear system to be physically realizable, it is necessary that the system respond only to past inputs. In equation form, this implies

$$h(\alpha, t) = 0 \quad \text{for} \quad \alpha < 0, \text{ independent of } t \tag{5.1}$$

Also, since there should be no response to inputs occurring in the infinite past,

$$\lim_{\alpha \to \infty} h(\alpha, t) = 0 \tag{5.2}$$

If $x(t)$ is an input to a time-varying linear system characterized by $h(\alpha, t)$, and if $y(t)$ is the output, then the output is given by

$$y(t) = \int_{-\infty}^{\infty} h(\alpha, t)\, x(t - \alpha)\, d\alpha \tag{5.3}$$

which is a weighted linear sum over the entire past history of the input. For a physically realizable system, the effective lower limit is zero since $h(\alpha, t) = 0$ for $\alpha < 0$.

$$x(t) \longrightarrow \boxed{\begin{array}{c}\text{Time-varying}\\ \text{linear system}\\ h(\alpha, t)\end{array}} \longrightarrow y(t)$$

Instead of using $h(\alpha, t)$, a time-varying linear system may be characterized by a <u>time-varying frequency response function</u> $\mathcal{H}(f, t)$ which is defined as the Fourier transform of $h(\alpha, t)$, namely,

$$\mathcal{H}(f, t) = \int_{-\infty}^{\infty} h(\alpha, t)\, e^{-j2\pi f\alpha}\, d\alpha \tag{5.4}$$

For a physically realizable system, the effective lower limit is zero. This quantity $\mathcal{H}(f, t)$ is a complex-valued function of f and t such that

$$\mathcal{H}(f, t) = \left|\mathcal{H}(f, t)\right| e^{j\phi(f, t)} \tag{5.5}$$

where the absolute value $\left|\mathcal{H}f(t)\right|$ measures the amplitude response of the system to a unit sinusoidal input function at frequency f, while $\phi(f)$ measures the corresponding phase shift. For a <u>constant parameter linear system</u> $\mathcal{H}(f, t)$ is replaced by $\mathcal{H}(f)$ since $h(\alpha, t)$ is replaced by $h(\alpha)$, and there is no dependence on t.

5.3 STABILITY AND REALIZABILITY

The system $h(\alpha, t)$ is said to be stable if every bounded input function produces a bounded output function. From Eq. (5.3),

$$|y(t)| = \left| \int_{-\infty}^{\infty} h(\alpha, t)\, x(t - \alpha)\, d\alpha \right| \leqq \int_{-\infty}^{\infty} |h(\alpha, t)|\, |x(t - \alpha)|\, d\alpha \tag{5.6}$$

When the input $x(t)$ is bounded, there exists some finite constant C such that

$$|x(t)| \leqq C \quad \text{for all } t \tag{5.7}$$

Now,

$$|y(t)| \leqq C \int_{-\infty}^{\infty} |h(\alpha, t)|\, d\alpha \tag{5.8}$$

Hence, if the time-varying weighting function $h(\alpha, t)$ is absolutely integrable, that is, if

$$\int_{-\infty}^{\infty} |h(\alpha, t)|\, d\alpha < \infty \tag{5.9}$$

then the output $y(t)$ will be bounded and the system will be stable. Thus, Eq. (5.9) is a sufficient condition for stability.

The condition of Eq. (5.9) is also a necessary condition for stability. The proof of this statement is quite simple and can be shown as follows: Suppose Eq. (5.9) is not satisfied, and suppose $x(t)$ is a bounded input defined by

$$\begin{aligned} x(-t) &= 1 \text{ for those } \alpha \text{ where } h(\alpha, 0) \geqq 0 \\ &= -1 \text{ for those } \alpha \text{ where } h(\alpha, 0) < 0 \end{aligned}$$

Now, the output

$$y(0) = \int_{-\infty}^{\infty} h(\alpha, 0)\, x(-\alpha)\, d\alpha = \int_{-\infty}^{\infty} |h(\alpha, 0)|\, d\alpha \rightarrow \infty$$

by hypothesis, and the system is not stable.

When considered in the extended complex frequency domain, the condition for physical realizability of a time-varying linear system takes an interesting form Let

$$H(s,t) = \int_{-\infty}^{\infty} h(a,t)\, e^{-sa}\, da \quad ; \quad s = \gamma + j2\pi f \tag{5.10}$$

By definition, the frequency response function

$$\mathcal{H}(f,t) = H(s,t) \qquad \text{when} \qquad \gamma = 0 \tag{5.11}$$

One can show, Ref. [2], that $h(a,t)$ will vanish for $a < 0$, (hence, the system is physically realizable) if the domain of convergence of $H(s,t)$ is at least a half plane $\gamma > 0$. When a system is stable, the domain of convergence $H(s,t)$ must include the imaginary axis $\gamma = 0$. Hence, if a system is both stable and physically realizable, then the domain of convergence for $H(s,t)$ is at least the entire right-half plane $\gamma \geqq 0$.

It should be observed that stability and realizability are independent properties. For example, a system defined by a weighting function $h(a,t) = e^{-|a|}$ is stable for

$$\int_{-\infty}^{\infty} |h(a,t)|\, da = \int_{-\infty}^{\infty} e^{-|a|} da = \frac{2}{a} < \infty$$

However, since $h(a,t)$ is not required to vanish for $a < 0$, this system is not physically realizable. The frequency function $H(s,t)$ for this example is given by

$$H(s,t) = \int_{-\infty}^{\infty} e^{-|a|} e^{-sa}\, da = \frac{2}{1 - s^2} \qquad \text{if } |s| < 1$$

which has a pole in the right half plane at the point $s = 1$, and hence is not realizable. If this $h(a,t)$ is made to vanish for $a < 0$, then $H(s,t)$ becomes $1/(s+1)$ with a pole in the left half plane at $s = -1$, and the system is now realizable.

5.4 BANDWIDTH PROPERTIES OF CONSTANT PARAMETER LINEAR SYSTEMS

Some bandwidth properties of constant parameter and time-varying linear systems, as developed in Ref. [4], are worthy of mention. For a constant parameter linear system, Eq. (5.3) becomes

$$y(t) = \int_{-\infty}^{\infty} h(a)\, x(t - a)\, da \tag{5.12}$$

where $h(a) = 0$ for $a < 0$ when the system is realizable. When $h(a) = \delta(a)$, a delta function,

$$x(t) = \int_{-\infty}^{\infty} \delta(a)\, x(t - a)\, da \tag{5.13}$$

The nth derivative of y(t) with respect to t is

$$y^{(n)}(t) = \int_{-\infty}^{\infty} h(a)\, x^{(n)}(t - a)\, da \tag{5.14}$$

Suppose x(t) is sinusoidal with frequency f. Then

$$x(t) = A \sin (2\pi f t + \phi) \tag{5.15}$$

where A and ϕ are constants. Now the second derivative

$$x^{(2)}(t) = -4\pi^2 A f^2\, x(t) \tag{5.16}$$

It follows that

$$y^{(2)}(t) = -4\pi^2 A f^2\, y(t) \tag{5.17}$$

Thus, y(t) is sinusoidal also with the same frequency f. This result shows that a constant parameter linear system cannot cause any frequency translation but can only modify the amplitude and phase of an input signal. In particular, there can be no bandwidth increases with such systems.

5.5 BANDWIDTH PROPERTIES OF TIME-VARYING LINEAR SYSTEMS

For time-varying linear systems, Eq. (5.3) states

$$y(t) = \int_{-\infty}^{\infty} h(\alpha, t)\, x(t - \alpha)\, d\alpha \tag{5.18}$$

where $h(\alpha, t) = 0$ for $\alpha < 0$ when the system is realizable. If an input $x(t)$ is sinusoidal with frequency f, it no longer follows that the output $y(t)$ will be sinusoidal with frequency f. The factor t in $h(\alpha, t)$ causes frequency translation to occur in addition to amplitude and phase changes.

Suppose $h(\alpha, t)$ is periodic in t with frequency F. Then, by expanding $h(\alpha, t)$ in a Fourier Series,

$$h(\alpha, t) = h_0(\alpha) + \sum_{n=1}^{\infty} h_n(\alpha, t) \tag{5.19}$$

where $h_n(\alpha, t)$ is sinusoidal in t with frequency nF. Now,

$$y(t) = y_0(t) + \sum_{n=1}^{\infty} y_n(t) \tag{5.20}$$

where

$$\begin{aligned} y_0(t) &= \int_{-\infty}^{\infty} h_0(\alpha)\, x(t - \alpha)\, d\alpha \\ y_n(t) &= \int_{-\infty}^{\infty} h_n(\alpha, t)\, x(t - \alpha)\, d\alpha \end{aligned} \tag{5.21}$$

The quantity $h_0(\alpha)$ represents a constant parameter linear system while $h_n(\alpha, t)$ represents a time-varying linear system.

A general form for $h_n(\alpha, t)$ is

$$h_n(\alpha, t) = A_n(\alpha) \cos \left[2\pi n F t + \phi_n(\alpha) \right] \tag{5.22}$$

Now,

$$
\begin{aligned}
y_n(t) &= \int_{-\infty}^{\infty} A_n(\alpha) \cos\left[2\pi Ft + \phi_n(\alpha)\right] x(t - \tau)\, d\alpha \\
&= \cos 2\pi nFt \int_{-\infty}^{\infty} A_n(\alpha)\left[\cos \phi_n(\alpha)\right] x(t - \alpha)\, d\alpha \\
&\quad - \sin 2\pi nFt \int_{-\infty}^{\infty} A_n(\alpha)\left[\sin \phi_n(\alpha)\right] x(t - \alpha)\, d\alpha
\end{aligned}
\tag{5. 23}
$$

For each sinusoidal component of frequency f in the input signal x(t), say, $x(t) = \sin 2\pi ft$, one can now verify that the output $y_n(t)$ will have sinusoidal components of frequencies $nF + f$ and $|nF - f|$. This follows from the trigonometric expression

$$
\begin{aligned}
&A(\alpha) \cos 2\pi Ft \sin 2\pi f(t - \alpha) - B(\alpha) \sin 2\pi nFt \sin 2\pi f(t - \alpha) \\
&\quad = \frac{A(\alpha)}{2} \cos 2\pi f\alpha\left[\sin 2\pi(nF + f)t - \sin 2\pi(nF - f)t\right] \\
&\quad - \frac{A(\alpha)}{2} \sin 2\pi f\alpha\left[\cos 2\pi(nF + f)t + \cos 2\pi(nF - f)t\right] \\
&\quad - \frac{B(\alpha)}{2} \cos 2\pi f\alpha\left[\cos 2\pi(nF - f)t - \cos 2\pi(nF + f)t\right] \\
&\quad + \frac{B(\alpha)}{2} \sin 2\pi f\alpha\left[\sin 2\pi(nF + f)t + \sin 2\pi(nF - f)t\right]
\end{aligned}
\tag{5. 24}
$$

Thus, a time-varying linear system produces in the output y(t) a higher bandwidth than the bandwidth of x(t). This result provides a theoretical explanation for some situations where an output noise bandwidth is greater than the input noise bandwidth.

5.6 LINEAR TRANSFORMATIONS OF RANDOM FUNCTIONS

An operator A is said to be linear if for any set of admissable functions $x_1, x_2, \ldots, x_n$ and constants $a_1, a_2, \ldots, a_n$, it follows that

$$A(a_1x_1 + a_2x_2 + \ldots + a_nx_n) = a_1A(x_1) + a_2A(x_2) + \ldots + a_nA(x_n) \tag{5.25}$$

The operator A can take many different forms. For a given value of x, it can determine a function y(x). For a given function x(t), it can determine a function y(t) as the result of carrying out a certain operation, e.g., differentiation. The process of integrating x(t) between definite limits generates a quantity which depends on the given function x(t) and on the integration limits.

An arbitrary linear transformation A_t of a random quantity x(t) may be represented symbolically by

$$y(z) = A_t\, x(t) \tag{5.26}$$

where the argument z may or may not be the same as t. In the case of differentiation z = t, while in the case of a definite integral z would be different from t.

For any linear operation, the process of averaging of the data (taking expected values) is commutative with the linear operation, namely,

$$E\left[y(z)\right] = E\left[A_t\, x(t)\right] = A_t\, E\left[x(t)\right] \tag{5.27}$$

where E denotes the expected value. This property will now be applied to moments of y(z) at different z.

The first moment of y(z) at $z = z_1$ is defined by

$$\mu_1^y(z_1) = E\left[y(z_1)\right] \tag{5.28}$$

which is the ensemble average of the process $\left\{y(z)\right\}$ at $z = z_1$. From Eq. (5.27), if t_1 corresponds to z_1,

$$\mu_1^y(z_1) = A_{t_1}\ \mu_1^x(t_1) \tag{5.29}$$

where $\mu_1^x(t_1) = E\left[x(t_1)\right]$.

The second moment is defined by the ensemble average of the squared values of $y(z)$ at $z = z_1$, namely,

$$\mu_2^y(z_1) = E\left[y^2(z_1)\right] \tag{5.30}$$

From Eqs. (5.26) and (5.27),

$$\mu_2^y(z_1) = E\left[A_{t_1}^2 \; x^2(t_1)\right] = A_{t_1}^2 \; \mu_2^x(t_1) \tag{5.31}$$

The mixed second moment is defined by the ensemble average

$$\mu_2^y(z_1, z_2) = E\left[y(z_1)\, y(z_2)\right]$$

If t_1 and t_2 correspond to z_1 and z_2, then Eq. (5.27) yields

$$\mu_2^y(z_1, z_2) = A_{t_1} \; A_{t_2} \; \mu_2^x(t_1, t_2) \tag{5.32}$$

where $\mu_2^x(t_1, t_2) = E\left[x(t_1)\, x(t_2)\right]$ This result can be extended to prove that n-fold moments of the image of x(t) under a linear transformation A_t equals the n-fold iteration of A_t applied to the n-fold moment of x(t). In equation form,

$$\mu_n^y(z_1, z_2, \ldots, z_n) = A_{t_1} A_{t_2} \cdots A_{t_n} \; \mu_n^x(t_1, t_2, \ldots, t_n) \tag{5.33}$$

where

$$\mu_n^y(z_1, z_2, \ldots, z_n) = E\left[y(z_1)\, y(z_2) \ldots y(z_n)\right] \tag{5.34}$$

These results will now be applied to arbitrary linear integral transformations of random functions. Time varying and constant parameter linear systems are merely special examples of such integral transformations.

5.7 INTEGRAL TRANSFORMATIONS OF RANDOM FUNCTIONS

An integral transformation of any particular random function $x(t)$ from an arbitrary random process $x(t)$ is defined by

$$F = \int_a^b x(t)\,\phi(t)\,dt \tag{5.35}$$

where $\phi(t)$ is an arbitrary given function for which the integral exists. For any given $\phi(t)$ and limits (a, b), the quantity F is a random variable which depends on the particular random function $x(t)$.

In order to investigate statistical properties of the random variable F, it is customary to break up the integration interval (a, b) into subintervals Δt and consider the approximation linear sum

$$F \approx \sum_{i=1}^{N} x(i\,\Delta t)\,\phi(i\,\Delta t)\,\Delta t \tag{5.36}$$

In general, the probability distribution followed by a sum of terms does not equal the probability distribution of the individual terms. If $x(t)$ follows a normal distribution, however, one can show that F will then be normally distributed. As a consequence of the central limit theorem, the random variable F will approximate a normal distribution if $x(t)$ is such that values arbitrarily near in time are mutually independent statistically. One can prove also that if F is considered as a superposition of a large number of small effects, none of which contributes significantly to the sum, then F will approximate a normal distribution for arbitrary $x(t)$.

Consider now the transformation of moments of $x(t)$ due to the linear integral operation. Since the average of a sum equals the sum of the averages, from Eq. (5.36), the first moment

$$\mu_1^F \approx \sum_{i=1}^{N} \mu_1^x(i\,\Delta t)\,\phi(i\,\Delta t)\,\Delta t \tag{5.37}$$

On passing to an integral, one obtains

$$\mu_1^F = \int_a^b \mu_1^x\,\phi(t)\,dt \tag{5.38}$$

This result can also be obtained directly from applying Eq. (5. 29) to Eq. (5. 35).

Similarly, by considering finite sums and passing to integral form, or directly from Eq. (5. 32) applied to Eq. (5. 35), one derives for the second moment

$$\mu_2^F = E\left[F^2\right] = \int_a^b \int_a^b \mu_2^x(t_1, t_2)\,\phi(t_1)\,\phi(t_2)\, dt_1\, dt_2 \qquad (5.39)$$

Eq. (5. 39) shows that the second moment of F requires knowledge of the mixed second moment of x(t). Extensions of Eq. (5. 39) to higher order moments of F of order n requires an n-fold integration of mixed nth order moments of x(t). For example,

$$\mu_3^F = E\left[F^3\right] = \int_a^b \int_a^b \int_a^b \mu_3^x(t_1, t_2, t_3)\,\phi(t_1)\,\phi(t_2)\,\phi(t_3)\, dt_1\, dt_2\, dt_3$$

5. 8 DERIVATIVE TRANSFORMATIONS OF RANDOM FUNCTIONS

The derivative of a random function x(t) from an arbitrary random process x(t) is defined by the linear operation

$$\lim_{\Delta t \to 0} \left[\frac{x(t+\Delta t) - x(t)}{\Delta t}\right] = x'(t) \qquad (5.40)$$

when this limit exists. Conditions for the existence of this limit will be found shortly. For any given x(t), the derivative x'(t) is a random variable which depends on x(t).

Consider the general class of problems where $\{x(t)\}$ follows a normal distribution with zero mean value for any t. Assume that there is a statistical dependence between values of $\{x(t)\}$ and $\{x(t+\Delta t)\}$ when Δt is sufficiently small. The difference $[x(t+\Delta t) - x(t)]$ as a linear function of two random variables will be normally distributed when x(t) and $x(t+\Delta t)$ are normally distributed. Assume that the joint probability density function of $x_1 = x(t)$ and $x_2 = x(t+\Delta t)$ is given by the joint normal distribution [1, p. 120],

$$p(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left[\frac{-(\sigma_2^2 x_1^2 - 2\rho\sigma_1\sigma_2 x_1 x_2 + \sigma_1^2 x_2^2)}{2\sigma_1^2\sigma_2^2(1-\rho^2)}\right] \tag{5.41}$$

where the mean values $\overline{x}_1$ and $\overline{x}_2$ equal zero and

$$\begin{aligned}
\sigma_1^2 &= \left\langle (x_1 - \overline{x})^2 \right\rangle = \left\langle x_1^2 \right\rangle = \left\langle x^2(t) \right\rangle = \sigma_1^2(t) \\
\sigma_2^2 &= \left\langle (x_2 - \overline{x}_2)^2 \right\rangle = \left\langle x_2^2 \right\rangle = \left\langle x^2(t+\Delta t) \right\rangle = \sigma_2^2(t+\Delta t) \\
\sigma_1\sigma_2\rho &= \left\langle (x_1 - \overline{x}_1)(x_2 - \overline{x}_2) \right\rangle = \left\langle x_1 x_2 \right\rangle = \left\langle x(t)\, x(t+\Delta t) \right\rangle = \sigma_1\sigma_2\,\rho(t, t+\Delta t)
\end{aligned} \tag{5.42}$$

As usual, the angular brackets denote ensemble averages. For nonstationary processes, the variance quantities $\sigma_1^2 = \sigma_1^2(t)$, $\sigma_2^2 = \sigma_2^2(t+\Delta t)$, and the normalized autocorrelation function $\rho = \rho(t, t+\Delta t)$ are functions of t and Δt. For stationary processes, $\sigma_1^2 = \sigma_2^2$, independent of t, and $\rho = \rho(\Delta t)$ alone.

The remainder of the discussion will prove that the derivative $x'(t)$ exists in the mean square sense if the (nonstationary) normalized autocorrelation function $\rho'(t,t) = 0$ where

$$\rho'(t,t) = \frac{d}{d(\Delta t)}\, \rho(t, t+\Delta t) \quad \text{at the point} \quad \Delta t = 0,\ t \text{ fixed} \tag{5.43}$$

Existence in the mean square sense is defined here by having a finite value for the variance of the quotient of Eq. (5.40) as $\Delta t \to 0$.

To prove this claim, make a change in variable

$$x_1 = x \quad ; \quad x_2 = x + \Delta x \tag{5.44}$$

This yields a new joint probability density function

$$p(x, x+\Delta x) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left[\frac{-(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2)x^2 + 2(\sigma_1^2 - \rho\sigma_1\sigma_2)x\,\Delta x}{2\sigma_1^2\sigma_2^2(1-\rho^2)}\right]$$

$$\text{times } \exp\left[\frac{-\sigma_1^2(\Delta x)^2}{2\sigma_1^2\sigma_2^2(1-\rho^2)}\right] \tag{5.45}$$

In order to compute the one-dimensional probability density function for Δx, it is necessary to evaluate

$$p(\Delta x) = \int_{-\infty}^{\infty} p(x, x + \Delta x)\, dx \tag{5.46}$$

After some manipulation, one can verify that Δx follows a normal distribution with zero mean value given by

$$p(\Delta x) = \frac{1}{\sqrt{2\pi}\,\sigma_{\Delta x}} \exp\left[\frac{-(\Delta x)^2}{2\sigma_{\Delta x}^2}\right] \tag{5.47}$$

where

$$\sigma_{\Delta x}^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2 \tag{5.48}$$

This result proves that the derivative of a normally distributed random variable will also follow a normal distribution since the derivative, Eq. (5.40), has the equivalent definition

$$x'(t) = \lim_{\Delta t \to 0} \left(\frac{\Delta x}{\Delta t}\right) \tag{5.49}$$

when this limit exists.

From Eq. (5.47), the variance of the quotient $(\Delta x/\Delta t)$ is given by

$$\sigma_{\Delta x/\Delta t}^2 = \frac{\sigma_{\Delta x}^2}{(\Delta t)^2} = \frac{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}{(\Delta t)^2} \tag{5.50}$$

where $\rho = \rho(t, t+\Delta t)$ is a function of t and Δt. As $\Delta t \to 0$, the numerator expression of Eq. (5.50) becomes $(\sigma_1 - \sigma_2)^2$ since $\rho(t, \tau) = 1$ for any t. To evaluate Eq. (5.50) in the limit, one should expand $\rho(t, t+\Delta t)$ in a power series about $\Delta t = 0$, holding t fixed, namely,

$$\rho(t, t+\Delta t) = 1 + \rho'(t, t)\Delta t + \frac{1}{2}\rho''(t, t)(\Delta t)^2 + \dots \tag{5.51}$$

Thus,

$$\sigma^2_{\Delta x/\Delta t} = \frac{(\sigma_1 - \sigma_2)^2 - 2\sigma_1\sigma_2\left[\rho'(t,t)\,\Delta t + \frac{1}{2}\,\rho''(t,t)(\Delta t)^2 + \ldots\right]}{(\Delta t)^2} \qquad (5.52)$$

The existence of the derivative $x'(t)$ in the mean square sense is defined by the existence of Eq. (5.52) as $\Delta t \to 0$. Now, as $\Delta t \to 0$, the standard deviation $\sigma_2 \to \sigma_1$. Thus, for small Δt,

$$\sigma^2_{\Delta x/\Delta t} \approx \frac{-2\sigma_1^2\left[\rho'(t,t)\;\Delta t + \frac{1}{2}\,\rho''(t,t)(\Delta t)^2 + \ldots\right]}{(\Delta t)^2}$$

One can now see that if $\rho'(t,t) \neq 0$, then Eq. (5.52) does not exist as $\Delta t \to 0$, and hence the derivative $x'(t)$ does not exist. However, if $\rho'(t,t) = 0$, then as $\Delta t \to 0$, Eq. (5.52) becomes

$$\sigma^2_{\Delta x/\Delta t} = -\sigma_1^2\,\rho''(t,t) \qquad (5.53)$$

where t is considered as a constant, and

$$\rho''(t,t) = \frac{d}{d(\Delta t)}\left[\frac{d}{d(\Delta t)}\;\rho(t,\;t + \Delta t)\right] \text{ at the point } \Delta t = 0 \qquad (5.54)$$

Now, the derivative expression of Eq. (5.49) will exist in the mean square sense. Since Eq. (5.53) must be positive, the second derivative $\rho''(t,t)$ must be negative. For stationary random processes, $\rho''(t,t) = \rho''(0)$, independent of t.

Consider next the transformation of moments of $x(t)$ due to the linear derivative operation. Since the operations of averaging and differentiation are interchangeable, the first order moment of the derivative

$$E\left[x'(t)\right] = E\left\{\lim_{\Delta t \to 0}\left[\frac{x(t + \Delta t) - x(t)}{\Delta t}\right]\right\}$$

$$= \lim_{\Delta t \to 0}\left\{\frac{E\left[x(t + \Delta t)\right] - E\left[x(t)\right]}{\Delta t}\right\} = \frac{d\,\mu_1^x(t)}{dt} \qquad (5.55)$$

where

$$\mu_1^x(t) = E\left[x(t)\right]$$

For a stationary random process, since $\mu_1(t)$ is independent of t, the first order moment of the derivative must equal zero.

Consider the mixed second-order moment of a joint derivative. Here, one obtains the extended result

$$E\left[x'(t_1)\,x'(t_2)\right] = \lim_{\substack{\Delta t_1 \to 0 \\ \Delta t_2 \to 0}} E\left\{\left[\frac{x(t_1+\Delta t_1) - x(t_1)}{\Delta t_1}\right]\left[\frac{x(t_2+\Delta t_2) - x(t_2)}{\Delta t_2}\right]\right\}$$

$$= \frac{\partial^2 \mu_2^x(t_1, t_2)}{\partial t_1\, \partial t_2} \tag{5.56}$$

where

$$\mu_2^x(t_1, t_2) = E\left[x(t_1)\,x(t_2)\right]$$

Extension of Eq. (5. 57) to higher order moments of n derivatives of $x(t)$ requires an n-fold partial differentiation of mixed nth order moments of $x(t)$. For example,

$$E\left[x'(t_1)\,x'(t_2)\,x'(t_3)\right] = \frac{\partial^3 \mu_3^x(t_1, t_2, t_3)}{\partial t_1\, \partial t_2\, \partial t_3}$$

Further discussion of these ideas and material in the preceding two sections appears in Ref. [3]. Many practical applications of these operations for electrical and mechanical systems are developed in this reference.

5.9 INPUT-OUTPUT RELATIONS FOR TIME-VARYING LINEAR SYSTEMS

Consider a nonstationary random process $\{x(t)\}$ acting as inputs to a time-varying linear system with weighting function $h(\alpha,t)$ and frequency response function $\mathcal{H}(f, t)$. For an arbitrary record $x(t)$ belonging to $\{x(t)\}$, the output $y(t)$ belonging to $\{y(t)\}$ is

$$y(t) = \int_{-\infty}^{\infty} h(\alpha, t)\, x(t - \alpha)\, d\alpha \tag{5.57}$$

It is clear that, in general, $\{\ddot{y}(t)\}$ will be a nonstationary random process since its statistical properties will be a function of t.

For a pair of times t_1, t_2, the product $y(t_1)y(t_2)$ is given by

$$y(t_1)y(t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\alpha, t_1)\, h(\beta, t_2)\, x(t_1 - \alpha)\, x(t_2 - \beta)\, d\alpha\, d\beta \tag{5.58}$$

where different variables of integration α, β are used to avoid confusion.

Upon taking ensemble averages, one obtains

$$R_y(t_1, t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\alpha, t_1)\, h(\beta, t_2)\, R_x(t_1 - \alpha,\, t_2 - \beta)\, d\alpha\, d\beta \tag{5.59}$$

This general result shows how an output nonstationary autocorrelation function may be determined from the input nonstationary autocorrelation function and the system weighting function. The operations in Eq. (5.59) all take place in a real-valued time domain.

By transforming to a complex-valued frequency domain, one can derive the equivalent relation for nonstationary power spectrum transformations,

$$S_y(f_3, f_4) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_x(f_1, f_2)\, \overline{J(f_1, f_1 - f_3)}\, J(f_2, f_2 - f_4)\, df_1\, df_2 \tag{5.60}$$

where

$$J(f_1, f_0) = \int_{-\infty}^{\infty} \mathcal{H}(f_1, t)\, e^{j2\pi f_0 t}\, dt \tag{5.61}$$

The quantity $\overline{J(f_1, f_0)}$ is the complex conjugate of $J(f_1, f_0)$. Equation (5.60) results from applying Eqs. (4.17), (4.19), and (5.4) to Eq. (5.59). For a physically realizable system, one uses the fact that $h(\alpha, t) = 0$ for $\alpha < 0$ so that the effective lower limits in Eq. (5.59) can be changed to $-\infty$ without changing the result.

If $S_x(f_1, f_2) = S_x(f_1)\, \delta(f_2 - f_1)$, a stationary process, then Eq. (5.60) becomes

$$S_y(f_3, f_4) = \int_{-\infty}^{\infty} S_x(f_2)\, \overline{J(f_2, f_2 - f_3)}\, J(f_2, f_2 - f_4)\, df_2 \tag{5.62}$$

When the system is a constant parameter linear system, where $\mathscr{H}(f_1, t) = \mathscr{H}(f_1)$ independent of t, Eq. (5.61) becomes

$$J(f_1, f_0) = \mathscr{H}(f_1) \int_{-\infty}^{\infty} e^{j2\pi f_0 t} \, dt = \mathscr{H}(f_1)\delta(f_0) \tag{5.63}$$

where $\delta(f_0)$ is the usual Dirac delta function. Now, Eq. (5.60) yields

$$S_y(f_3, f_4) = \overline{\mathscr{H}(f_3)}\mathscr{H}(f_4) S_x(f_3, f_4) \tag{5.64}$$

where

$$\mathscr{H}(f) = \int_{-\infty}^{\infty} h(\alpha) e^{-j2\pi f\alpha} \, d\alpha \tag{5.65}$$

For the special case of stationary random processes where $\mathscr{S}_x(f_3, f_4)$ is nonzero only for $f = f_3 = f_4$, Eq. (5.64) reduces to the familiar result

$$S_y(f) = |\mathscr{H}(f)|^2 S_x(f) \tag{5.66}$$

The change in form exhibited by Eqs. (5.60), (5.62), (5.64), and (5.66) is worthy of note.

Further interesting input-output relations can be found by examining the nonstationary cross-correlation function, and nonstationary cross-power spectral density function between the input process $\{x(t)\}$ and the output process $\{y(t)\}$. From Eq. (5.57), the product of $x(t_1)$ by $y(t_2)$ is

$$x(t_1) y(t_2) = \int_{-\infty}^{\infty} h(\beta, t_2) x(t_1) x(t_2 - \beta) \, d\beta$$

The taking of ensemble averages now yields

$$R_{xy}(t_1, t_2) = \int_{-\infty}^{\infty} h(\beta, t_2) R_x(t_1, t_2 - \beta) \, d\beta \tag{5.67}$$

Equation (5.67) is a general result in the real-valued time domain between the nonstationary input autocorrelation function, the time-varying linear system weighting function, and the nonstationary input-output cross-correlation function.

When Eq. (5.67) is transformed to a complex-valued frequency domain, there results from Eqs. (4.20) and (5.4) after a certain amount of straightforward manipulation the nonstationary cross-power spectrum relation

$$S_{xy}(f_3, f_4) = \int_{-\infty}^{\infty} S_x(f_3, f_2)\, J(f_2, f_2 - f_4)\, df_2 \tag{5.68}$$

The quantity J is defined by Eq. (5.61). Observe that the form of Eq. (5.68) is simpler than the form of Eq. (5.60).

If $S_x(f_3, f_2) = S_x(f_3)\delta(f_3 - f_2)$, a stationary process, then Eq. (5.68) becomes

$$S_{xy}(f_3, f_4) = S_x(f_3)\, J(f_3, f_3 - f_4) \tag{5.69}$$

For the important class of constant parameter linear systems, Eq. (5.68) becomes with the aid of Eq. (5.63),

$$S_{xy}(f_3, f_4) = \mathcal{H}(f_4)\, S_x(f_3, f_4) \tag{5.70}$$

Upon further reduction to the special case of stationary random processes, Eq. (5.70) becomes the stationary cross-power spectrum relation

$$S_{xy}(f) = \mathcal{H}(f)\, S_x(f) \tag{5.71}$$

This result, like Eq. (5.66) is well-known and provides a required check on this work.

5.10 REFERENCES

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2. James, H. M., N. B. Nichols, and R. S. Phillips, Theory of Servomechanisms, MIT Rad. Lab. Series, vol. 25, McGraw-Hill Book Co., 1947. Chapter 2.

3. Lebedev, V. L., Random Processes in Electrical and Mechanical Systems, (trans. from Russian), Office of Technical Services, U. S. Dept. of Commerce, Washington, D. C. 1961 (PST Cat. No. 200).

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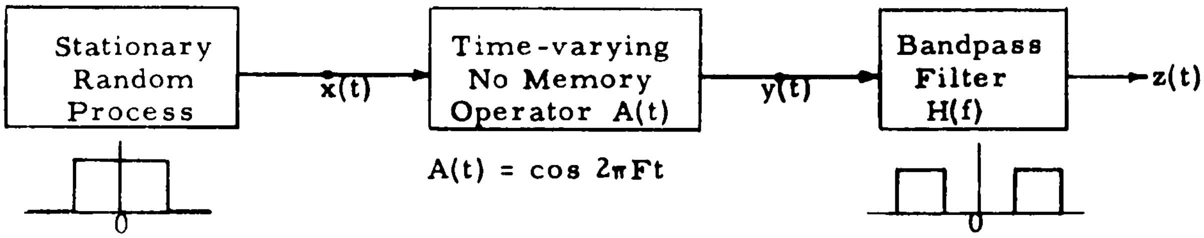
6. EXAMPLE

NONSTATIONARY PROCESS RESULTING FROM AMPLITUDE MODULATED AND FILTERED STATIONARY PROCESS

The following example is taken from Ref. [1] of Section 4. The notation has been changed to agree with the present material, and additional expressions have been calculated to illustrate special formulas that have been derived in Sections 4 and 5 which do not appear in Ref. [1]. Knowledge of the spectral changes occurring in this system may be useful to designers of spectrum simulators, in addition to helping to explain certain observed physical phenomena.



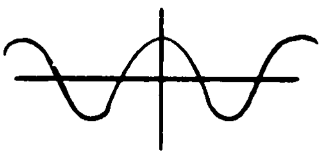
(1) Assume that the amplitude modulated time-varying output

$$y(t) = \int_{-\infty}^{\infty} a(\alpha, t)\, x(t - \alpha)\, d\alpha = A(t)\, x(t) \tag{6.1}$$

Hence the time-varying weighting function

$$a(\alpha, t) = A(t)\delta(\alpha) \tag{6.2}$$

where $\delta(\alpha)$ is a delta function. Note that $a(\alpha, t) = 0$ for $\alpha < 0$ so that this time-varying operator is physically realizable. In fact, the output $y(t)$ depends only upon the input $x(t)$ at time t and is not effected by any of its past behavior. For this example, let $A(t)$ be a cosine wave

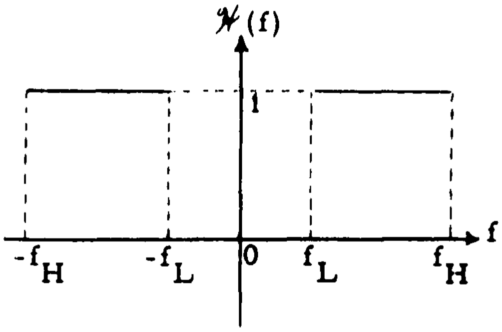


$$A(t) = \cos 2\pi F t \quad \text{where} \quad F = \text{constant} \tag{6.3}$$

(2) Assume that the filtered constant parameter output

$$z(t) = \int_{-\infty}^{\infty} h(\alpha)\, y(t - \alpha)\, d\alpha \tag{6.4}$$

where $h(\alpha)$ is described by its frequency response function $\mathcal{H}(f)$ as an ideal bandpass filter.



$$\mathcal{H}(f) = \int_{-\infty}^{\infty} h(\alpha)\, e^{-j2\pi f\alpha}\, d\alpha$$
$$= 1 \qquad f_L \leq |f| \leq f_H \tag{6.5}$$
$$= 0 \qquad \text{otherwise}$$

Now, from the inverse to Eq. (5.65), the constant parameter weighting function of a bandpass filter is given by

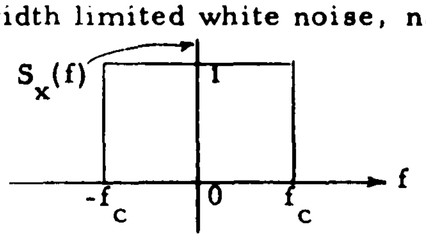
$$h(\alpha) = \int_{-\infty}^{\infty} \mathcal{H}(f)\, e^{j2\pi f\alpha}\, df$$
$$= \frac{2}{\pi\alpha} \sin\left[\pi(f_H - f_L)\,\alpha\right] \cos\left[\pi(f_L + f_H)\,\alpha\right] \tag{6.6}$$

The case of a lowpass filter where $f_L = 0$ is given by

$$h(\alpha) = \frac{2}{\pi\alpha} \sin \pi f_H \alpha \cos \pi f_H \alpha = \frac{\sin 2\pi f_H \alpha}{\pi\alpha} \tag{6.7}$$

Note that $h(\alpha) \neq 0$ for $\alpha < 0$ so that constant parameter bandpass or lowpass filters are not physically realizable unless this restriction is imposed.

(3) Assume that the stationary random process $x(t)$ has zero mean value and has a constant power spectral density function corresponding to bandwidth limited white noise, namely



$$S_x(f) = 1 \quad ; \quad |f| \leq f_c$$
$$= 0 \qquad \text{otherwise} \tag{6.8}$$

The corresponding stationary autocorrelation function from Eq.(4.29) is

$$R_x(\tau) = \int_{-\infty}^{\infty} S_x(f)\, e^{j2\pi f\tau}\, df = \frac{\sin 2\pi f_c \tau}{\pi\tau} \tag{6.9}$$

which is the same form as the weighting function of a lowpass filter. Note that the variance $R_x(0) = 2f_c$.

Problem: Compute the various stationary and nonstationary correlation functions and spectral density functions corresponding to the output records $\{y(t)\}$ and $\{z(t)\}$, individually, pairwise, and relative to the input records $\{x(t)\}$.

1. Properties of $\{y(t)\}$

First of all, consider the intermediate random process $\{y(t)\}$. From Eqs. (4.8), (6.1), and (6.3) the autocorrelation function

$$
\begin{aligned}
R_y(t_1, t_2) = \langle y(t_1)\, y(t_2)\rangle &= A(t_1)\, A(t_2)\, R_x(t_1, t_2) \\
&= \cos 2\pi F t_1 \cos 2\pi F t_2\, R_x(t_2 - t_1) \\
&= \frac{1}{2}\left[\cos 2\pi F(t_1 + t_2) + \cos 2\pi F(t_2 - t_1)\right] R_x(t_2 - t_1) \qquad (6.10)
\end{aligned}
$$

where $R_x(\tau) = R_x(t_2 - t_1)$ is given by Eq. (6.9). From Eq. (4.33)

$$\mathcal{R}_y(t, \tau) = \frac{1}{2}\left[\cos 4\pi F t + \cos 2\pi F \tau\right] R_x(\tau) \qquad (6.11)$$

Thus, y(t) is a nonstationary random process since its autocorrelation function is not independent of t. Note that $R_y(t, \tau)$ is locally stationary since it is separable into the product of functions of t and τ, namely

$$\mathcal{R}_y(t, \tau) = R_{1A}(t)\, R_{2A}(\tau) + R_{1B}(t)\, R_{2B}(\tau) \qquad (6.12)$$

where

$$
\begin{aligned}
R_{1A}(t) &= \frac{1}{2}\cos 4\pi F t \quad ; \quad R_{2A}(\tau) = R_x(\tau) \\
R_{1B}(t) &= \frac{1}{2} \quad ; \quad R_{2B}(\tau) = \left[\cos 2\pi F \tau\right] R_x(\tau)
\end{aligned} \qquad (6.13)
$$

The mean value of y(t) is zero when the mean value of x(t) is zero because of the linear operation involved. Hence the variance of y(t) is given by $\mathcal{R}_y(t, 0) = (1/2)(1 + \cos 4\pi F t)$.

From Equations (4.40), (4.41), and (4.42), the nonstationary power spectral density function corresponding to Eq. (6.12) is

$$\mathcal{S}_y(f, g) = S_{1A}(f)\, S_{2A}(g) + S_{1B}(f)\, S_{2B}(g) \tag{6.14}$$

where

$$\begin{aligned} S_{1A}(f) &= \frac{1}{4}\Big[\delta(f - 2F) + \delta(f + 2F)\Big] \quad ; \quad S_{2A}(g) = S_x(g) \\ S_{1B}(f) &= \frac{1}{2}\delta(f) \quad ; \quad S_{2B}(g) = \frac{1}{2}\Big[S_x(g - F) + S_x(g + F)\Big] \end{aligned} \tag{6.15}$$

Relative to the (f_1, f_2) plane, from Eqs. (4.35),

$$S_y(f_1, f_2) = \mathcal{S}_y\left[(f_2 - f_1), \left(\frac{f_1 + f_2}{2}\right)\right] \tag{6.16}$$

Equations (6.14) and (6.15) show that the function $\mathcal{S}_y(f, g)$ exists only along the lines $f = 0$ and $f = \pm\, 2F$ in the (f, g) plane. Consequently, since $f = f_2 - f_1$, the function $S_y(f_1, f_2)$ exists only along the lines $f_2 = f_1$ and $f_2 = f_1 \pm 2F$ in the (f_1, f_2) plane.

2. Joint Properties of $\{x(t)\}$ and $\{y(t)\}$

Consider next the pair of random processes $\{x(t)\}$ and $\{y(t)\}$. From Eq. (5.69), since $x(t)$ is stationary, the nonstationary cross-power spectral density function $S_{xy}(f_3, f_4)$ is given by

$$S_{xy}(f_3, f_4) = S_x(f_3)\, J(f_3, f_3 - f_4) \tag{6.17}$$

where

$$J(f_1, f_0) = \int_{-\infty}^{\infty} A(f_1, t)\, e^{j2\pi f_0 t}\, dt \tag{6.18}$$

For the present example, from Eqs. (6. 2) and (6. 3),

$$A(f_1, t) = A(t) = \cos 2\pi Ft \tag{6. 19}$$

Hence

$$J(f_1, f_0) = J(f_0) = \int_{-\infty}^{\infty} \cos 2\pi Ft\, e^{j2\pi f_0 t}\, dt$$

$$= \frac{1}{2}\left[\delta(f_0 - F) + \delta(f_0 + F)\right] \tag{6. 20}$$

Equations (6. 17) and (6. 20) now yield

$$S_{xy}(f_3, f_4) = \frac{1}{2} S_x(f_3)\left[\delta(f_4 - f_3 + F) + \delta(f_4 - f_3 - F)\right] \tag{6. 21}$$

Thus, the delta functions in Eq. (6. 21) show that the function $S_{xy}(f_3, f_4)$ exists only along the lines $f_4 = f_3 \pm F$ in the (f_3, f_4) plane. From Eq. (4. 35), by a change in variables to the (f, g) plane, one derives

$$\mathcal{S}_{xy}(f, g) = S_{xy}(g - \frac{f}{2}, g + \frac{f}{2}) = \frac{1}{2} S_x(g - \frac{f}{2})\left[\delta(f + F) + \delta(f - F)\right] \tag{6. 22}$$

$$= \frac{1}{2} S_x(g + \frac{F}{2})\,\delta(f + F) + \frac{1}{2} S_x(g - \frac{F}{2})\,\delta(f - F) \tag{6. 23}$$

The nonstationary cross-correlation function is given by

$$R_{xy}(t_3, t_4) = \langle x(t_3)\, y(t_4)\rangle = \langle x(t_3)\, A(t_4)\, x(t_4)\rangle$$

$$= A(t_4)\, R_x(t_4 - t_3) = \left[\cos 2\pi Ft_4\right] R_x(t_4 - t_3) \tag{6. 24}$$

From Eq. (4.33) this is equivalent to

$$\mathcal{R}_{xy}(t, \tau) = \cos 2\pi F(t - \frac{\tau}{2})\, R_x(\tau)$$

$$= \cos 2\pi Ft\left[R_x(\tau) \cos \pi F\tau\right] + \sin 2\pi Ft\left[R_x(\tau) \sin \pi Ft\right] \tag{6. 25}$$

Thus $R_{xy}(t, \tau)$ is of a separable form. The cross-covariance $\mathcal{R}_{xy}(t,0) = \cos 2\pi Ft$.

Transformation of Eq. (6. 25) as was done previously in obtaining Eqs. (6. 14) and (6. 15) will yield (6. 23), providing a necessary check on the work.

An alternative method which gives another check on the validity of Eqs. (6. 14) and (6. 15) can be obtained by substituting Eq. (6. 20) into Eq. (5. 62). This yields

$$S_y(f_3, f_4) = \frac{1}{4}\int_{-\infty}^{\infty} S_x(f_2)\left[\delta(f_2 - f_3 - F) + \delta(f_2 - f_3 + F)\right]\left[\delta(f_2 - f_4 - F) + \delta(f_2 - f_4 + F)\right] df_2$$

$$= \frac{1}{4}\Big[S_x(f_4 + F)\,\delta(f_4 - f_3) + S_x(f_3 + F)\,\delta(f_4 - f_3 - 2F) + S_x(f_3 - F)\,\delta(f_4 - f_3 + 2F) + S_x(f_4 - F)\,\delta(f_4 - f_3)\Big] \tag{6. 26}$$

In deriving Eq. (6. 26), there is need of the fact that $S_x(f)$ and $\delta(f)$ are both even functions of f. The reason for writing Eq. (6. 26) in this form is to bring out the fact that $S_y(-f_3, -f_4) = S_y(f_3, f_4)$. Now, from Eq. (4.35), by substituting $f_3 = g - \frac{f}{2}$, $f_4 = g + \frac{f}{2}$, one obtains

$$\mathcal{S}_y(f, g) = S_y(g - \tfrac{f}{2}, g + \tfrac{f}{2})$$

$$= \frac{1}{4}\Big[S_x(g + \tfrac{f}{2} + F)\;\delta(f) + S_x(g - \tfrac{f}{2} + F)\;\delta(f - 2F) + S_x(g - \tfrac{f}{2} - F)\;\delta(f + 2F) + S_x(g + \tfrac{f}{2} - F)\;\delta(f)\Big] \tag{6. 27}$$

$$= \frac{1}{4}\Big[S_x(g + F)\,\delta(f) + S_x(g)\,\delta(f - 2F) + S_x(g)\;\delta(f + 2F) + S_x(g - F)\,\delta(f)\Big] \tag{6. 28}$$

which is precisely Eq. (6. 14).

A check on Eq.(6. 24) is provided by substituting Eq.(6. 21) into Eq. (4. 20). The result is

$$R_{xy}(t_3, t_4) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_{xy}(f_3, f_4)\, e^{-j2\pi(f_3 t_3 - f_4 t_4)}\, df_3\, df_4$$

$$= \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_x(f_3)\left[\delta(f_4 - f_3 + F) + \delta(f_4 - f_3 - F)\right] e^{-j2\pi(f_3 t_3 - f_4 t_4)}\, df_4\, df_3$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} S_x(f_3)\, e^{j2\pi f_3(t_4 - t_3)}\left[e^{j2\pi F t_4} + e^{j2\pi F t_4}\right] df_3$$

$$= R_x(t_4 - t_3)\cos 2\pi F t_4$$

using Eq. (4. 29) to obtain the $R_x(t_4 - t_3)$ term.

Similarly, one can derive Eq(6. 10) from Eq. (6. 26) by substituting Eq. (6. 26) into Eq. (4. 16) and carrying out the various manipulations that are required.

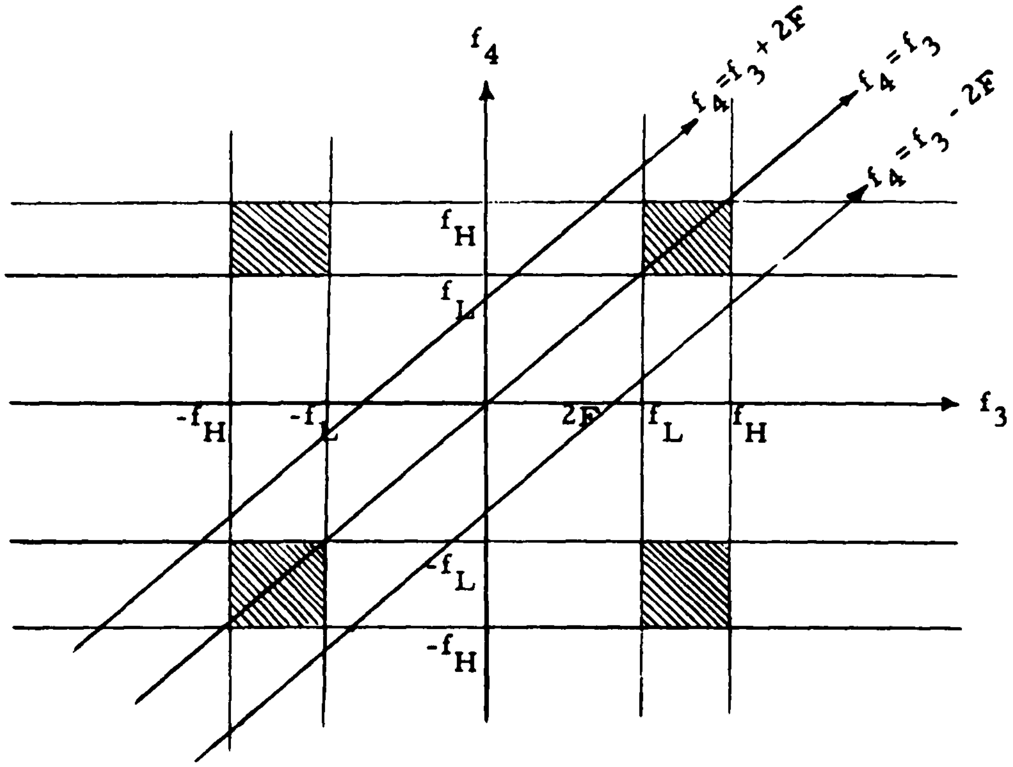
3. Properties of z(t)

The problem of determining properties of the final output process $\{z(t)\}$ is more difficult because of the filtering effects and will now be considered. From Eq. (5.64), the nonstationary power spectral density function $S_z(f_3, f_4)$ is given by

$$S_z(f_3, f_4) = \overline{\mathcal{H}(f_3)}\mathcal{H}(f_4)\, S_y(f_3, f_4) \qquad (6.29)$$

For this example, $S_y(f_3, f_4)$ is given by Eq. (6.26), whereupon it follows that $S_z(f_3, f_4)$ exists only along the lines $f_4 = f_3$ and $f_4 = f_3 \pm 2F$ in the (f_3, f_4) plane.

From Eq. (6.5), the frequency response function $\mathcal{H}(f)$ equals unity only for $f_L \le |f| \le f_H$, and is zero elsewhere. Hence the product $\overline{\mathcal{H}(f_3)}\mathcal{H}(f_4)$ equals unity only within the four squares in the (f_3, f_4) plane bounded by $f_L \le |f_1| \le f_H$, $f_L \le |f_2| \le f_H$, and is zero elsewhere. See sketch.



From Eq. (6.29), it now follows that the function $S_z(f_3, f_4)$ is equal to zero except along the portion of the lines $f_4 = f_3$ and $f_4 = f_3 \pm 2F$ which lie inside one of these four squares. In particular, $f_4 = f_3$ will always cross the two squares in the first and third quadrants, while $f_4 = f_3 \pm 2F$ will cross these same two squares if and only if $2F < (f_H - f_L)$. There is no crossing of the two squares in the second and fourth quadrants unless $2F > f_H$. It will be assumed that $2F \leqq f_H$, so that only the crossings in the first and third quadrants need be considered.

Thus, if $f_H \geqq 2F \geqq (f_H - f_L)$, from Eqs. (6.26) and (6.29),

$$S_z(f_3, f_4) = \frac{1}{4}\left[S_x(f_4 + F)\,\delta(f_4 - f_3) + S_x(f_4 - F)\,\delta(f_4 - f_3)\right] \qquad (6.30)$$

while if $2F < (f_H - f_L)$, then in addition to the above,

$$S_z(f_3, f_4) = \frac{1}{4}\left[S_x(f_3 + F)\,\delta(f_4 - f_3 - 2F) + S_x(f_3 - F)\,\delta(f_4 - f_3 + 2F)\right] \qquad (6.31)$$

Note that $S_z(-f_3, -f_4) = S_z(f_3, f_4)$.

From Eq. (6.8), the terms $S_x(f_4 + F)$ and $S_x(f_4 - F)$ are zero for all f_4 except when

$$\begin{aligned} S_x(f_4 + F) &= 1 \quad ; \quad -f_c - F \leq f_4 \leq f_c - F \\ S_x(f_4 - F) &= 1 \quad ; \quad -f_c + F \leq f_4 \leq f_c + F \end{aligned} \qquad (6.32)$$

Assume that $f_c \geqq f_4 + F$ so that the terms of Eq.(6.32) are positive inside the square bounded by $f_L \leqq f_3 \leqq f_H$, $f_L \leqq f_4 \leqq f_H$. A similar condition exists for $S_x(f_3 \pm F)$ when $f_c \geqq f_3 + F$.

By Eq. (4. 16) and the remarks following Eq. (6. 29), the autocorrelation function $R_z(t_3, t_4)$ in the present example is now given by

$$R_z(t_3, t_4) = \left[\int_{-\infty}^{0} \int_{-\infty}^{0} + \int_{0}^{\infty} \int_{0}^{\infty} \right] S_z(f_3, f_4)\, e^{-j2\pi(f_3 t_3 - f_4 t_4)}\, df_3\, df_4$$

$$= 2 \int_{f_L}^{f_H} \int_{f_L}^{f_H} S_z(f_3, f_4) \cos 2\pi(f_4 t_4 - f_3 t_3)\, df_3\, df_4 \qquad (6.33)$$

since $S_z(-f_3, -f_4) = S_z(f_3, f_4)$

If $f_H \geqq 2F \geqq (f_H - f_L)$, then substitution from Eqs. (6. 30) and (6. 32) yields

$$R_z(t_3, t_4) = \int_{f_L}^{f_H} \int_{f_L}^{f_H} \delta(f_4 - f_3) \cos 2\pi(f_4 t_4 - f_3 t_3)\, df_3\, df_4 \qquad (6.34)$$

On setting $f = f_4 - f_3$, $df = -df_3$, one obtains

$$R_z(t_3, t_4) = \int_{f_L}^{f_H} \int_{f_4 - f_H}^{f_4 - f_L} \delta(f) \cos 2\pi\left[f_4(t_4 - t_3) + f t_3\right] df\, df_4$$

$$= \int_{f_L}^{f_H} \cos 2\pi f_4(t_4 - t_3)\, df_4$$

$$= (f_H - f_L) \cos\left[\pi(f_H + f_L)(t_4 - t_3)\right] \frac{\sin\left[\pi(f_H - f_L)(t_4 - t_3)\right]}{\pi(f_H - f_L)(t_4 - t_3)} \qquad (6.35)$$

Since $R_z(t_3, t_4)$ is a function only of $(t_4 - t_3)$, the output process $\{z(t)\}$ is stationary here.

The mean value of $z(t)$ is zero when the mean value of $y(t)$ is zero because of the linear nature of the bandpass filter. Hence the variance of $z(t)$ is given by $R_z(t, t)$.

From Eq. (6. 35),

$$R_z(t, t) = (f_H - f_L) \quad \text{when} \quad f_H \geqq 2F \geqq (f_H - f_L) \qquad (6.36)$$

which is seen to be independent of t here.

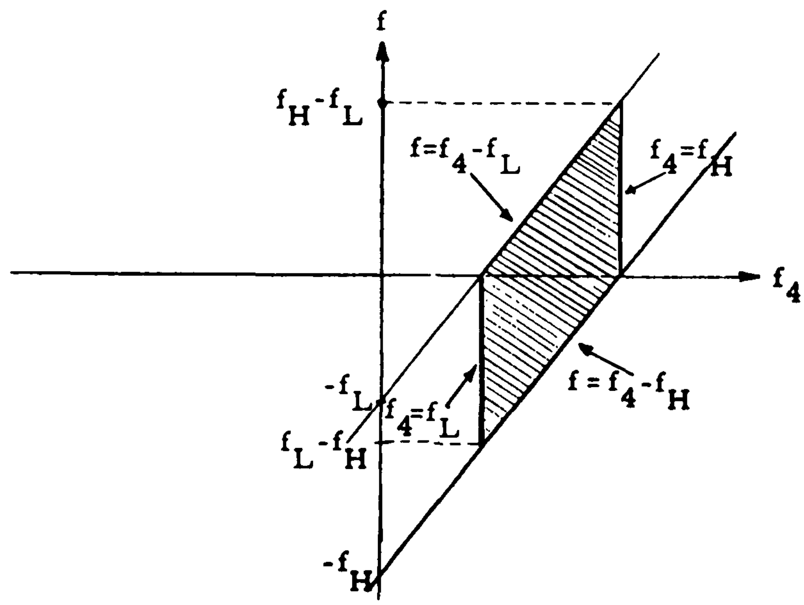
If $2F < (f_H - f_L)$, then, from Eq. (6. 31), in addition to Eq. (6. 35), the autocorrelation function $R_z(t_3, t_4)$ includes the term

$$R_z(t_3, t_4) = \frac{1}{2}\int_{f_L}^{f_H}\int_{f_L}^{f_H}\left[\delta(f_4 - f_3 + 2F) + \delta(f_4 - f_3 - 2F)\right]\cos 2\pi(f_4t_4 - f_3t_3)\,df_3\,df_4$$

$$(6.37)$$

$$= \frac{1}{2}\int_{f_L}^{f_H}\int_{f_4-f_H}^{f_4-f_L}\left[\delta(f + 2F) + \delta(f - 2F)\right]\cos 2\pi\left[f_4(t_4 - t_3) + ft_3\right]df\,df_4$$

on setting $f = f_4 - f_3$, $df = -df_3$. In order to interpret and evaluate this expression properly, it is now necessary to change the order of integration between f and f_4. The sketch below shows how to find the new limits, and shows that for this case one must break up the integration into two parts when the limits are reversed.



Thus, the result of changing the order of integration gives a pair of double integrals, namely,

$$R_z(t_3, t_4) = A_z(t_3, t_4) + B_z(t_3, t_4) \tag{6.38}$$

where

$$A_z(t_3, t_4) = \frac{1}{2} \int_{f_L - f_H}^{0} \int_{f_L}^{f_H + f} \left[\delta(f + 2F) + \delta(f - 2F)\right] \cos 2\pi\left[f_4(t_4 - t_3) + ft_3\right] df_4 \, df$$

$$\tag{6.39}$$

$$B_z(t_3, t_4) = \frac{1}{2} \int_{0}^{f_H - f_L} \int_{f_L + f}^{f_H} \left[\delta(f + 2F) + \delta(f - 2F)\right] \cos 2\pi\left[f_4(t_4 - t_3) + ft_3\right] df_4 \, df$$

If desired, one can now carry out these integrations for the general case of arbitrary t_3 and t_4. The result will be fairly complicated.

For the special point where $t_3 = t_4 = t$, one obtains for the variance of $z(t)$, in addition to Eq. (6.36) the term

$$R_z(t, t) = A_z(t, t) + B_z(t, t) \tag{6.40}$$

where

$$A_z(t, t) = \frac{1}{2} \int_{f_L - f_H}^{0} \int_{f_L}^{f_H + f} \left[\delta(f + 2F) + \delta(f - 2F)\right] \cos 2\pi f t_3 \, df_4 \, df$$

$$= \frac{1}{2} \int_{f_L - f_H}^{0} (f_H - f_L + f) \, \delta(f + 2F) \cos 2\pi ft \, df$$

$$= \frac{1}{2} (f_H - f_L - 2F) \cos 4\pi Ft \tag{6.41}$$

and

$$B_z(t,t) = \frac{1}{2}\int_0^{f_H - f_L}\int_{f_L + f}^{f_H}\left[\delta(f + 2F) + \delta(f - 2F)\right]\cos 2\pi f t_3 \, df_4 \, df$$

$$= \frac{1}{2}\int_0^{f_H - f_L}(f_H - f_L - f)\,\delta(f - 2F)\cos 2\pi f t \, df$$

$$= \frac{1}{2}(f_H - f_L - 2F)\cos 4\pi F t \tag{6.42}$$

Hence, the additional variance when $2F < (f_H - f_L)$ is given by

$$R_z(t,t) = (f_H - f_L - 2F)\cos 4\pi F t \tag{6.43}$$

Since this term is not independent of t, Eq. (6. 43) proves that the autocorrelation function is nonstationary when $2F < f_H - f_L$. Note that Eq. (6. 43) becomes zero when $2F = (f_H - f_L)$. The complete variance for cases when $2F < (f_H - f_L)$ is the sum of Eqs.(6. 36) and (6. 43), namely,

$$R_{zz}(t,t) = (f_H - f_L) + (f_H - f_L - 2F)\cos 4\pi F t \tag{6.44}$$

when $2F < (f_H - f_L)$.

4. Joint Properties of $\{x(t)\}$ and $\{z(t)\}$

Substitution of Eq. (6. 1) into Eq. (6. 4) gives

$$z(t) = \int_{-\infty}^{\infty} \left[h(a)\, A(t - a)\right] x(t - a)\, da \qquad (6.45)$$

which shows how $z(t)$ is related to $x(t)$. From Eq. (5. 69) the non-stationary cross-power spectral density function $S_{xz}(f_3, f_4)$ is given by

$$S_{xz}(f_3, f_4) = S_x(f_3)\, J(f_3,\, f_3 - f_4) \qquad (6.46)$$

where

$$J(f_1, f_0) = \int_{-\infty}^{\infty} \mathcal{H}(f_1, t)\, e^{j2\pi f_0 t}\, dt \qquad (6.47)$$

It follows from Eq. (6. 45) that

$$\mathcal{H}(f_1, t) = \int_{-\infty}^{\infty} \left[h(a)\, A(t - a)\right] e^{-j2\pi f_1 a}\, da \qquad (6.48)$$

From Eqs. (6. 3) and (6. 5), using a complex form for cos t,

$$\mathcal{H}(f_1, t) = \frac{1}{2}\int_{-\infty}^{\infty}\left[h(a)\; e^{j2\pi F(t-a)} + e^{-j2\pi F(t-a)}\right] e^{-j2\pi f_1 a}\, da$$

$$= \frac{1}{2}\left[e^{j2\pi Ft}\,\mathcal{H}(f_1 + F) + e^{-j2\pi Ft}\,\mathcal{H}(f_1 - F)\right] \qquad (6.49)$$

Substitution of Eq. (6. 49) into Eq. (6. 47) yields

$$J(f_1, f_0) = \frac{1}{2}\left[\mathcal{H}(f_1 + F)\delta(f_0 + F) + \mathcal{H}(f_1 - F)\delta(f_0 - F)\right] \qquad (6.50)$$

Hence, from Eqs. (6.46) and (6.50),

$$S_{xz}(f_3, f_4) = \frac{1}{2} S_x(f_3)\left[\mathcal{H}(f_3 + F)\delta(f_4 - f_3 - F) + \mathcal{H}(f_3 - F)\delta(f_4 - f_3 + F)\right] \tag{6.51}$$

From Eq. (6.5), the terms $\mathcal{H}(f_3 + F)$ and $\mathcal{H}(f_3 - F)$ are zero except when

$$\begin{aligned} \mathcal{H}(f_3 + F) &= 1 \quad ; \quad f_L \leq |f_3 + F| \leq f_H \\ \mathcal{H}(f_3 - F) &= 1 \quad ; \quad f_L \leq |f_3 - F| \leq f_H \end{aligned} \tag{6.52}$$

The delta functions in Equation (6.51) now show that $S_{xz}(f_3, f_4)$ exists only along the line $f_4 = f_3 + F$ in the (f_3, f_4) plane where $\mathcal{H}(f_3 + F) = 1$, and the line $f_4 = f_3 - F$ in the (f_3, f_4) plane where $\mathcal{H}(f_3 - F) = 1$. Note that $S_{xz}(-f_3, -f_4) = S_{xz}(f_3, f_4)$. If $F \leqq f_L$, then in the first quadrant, $\mathcal{H}(f_3 + F) = 1$ when $f_L - F \leq f_3 \leq f_H - F$ and $\mathcal{H}(f_3 - F) = 1$ when $f_L + F \leq f_3 \leq f_H + F$.

Assume that $f_c \geqq (f_H + F)$ so that $S_x(f_3)$ will be unity inside the regions covered by Eq. (6.52). From Eqs. (4.16), (6.51), and (6.52), the cross-correlation function $R_{xz}(t_3, t_4)$, assuming $F \leqq f_L$, is now given by

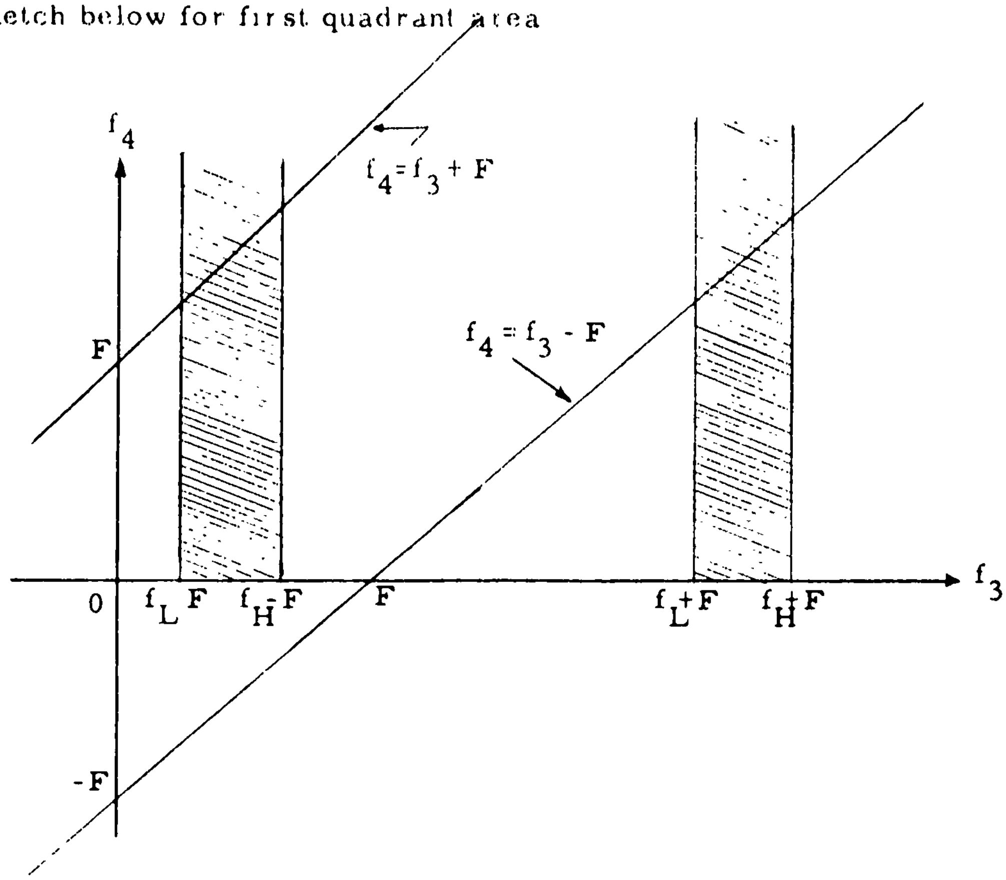
$$R_{xz}(t_3, t_4) = 2\int_0^{\infty}\int_0^{\infty} S_{xz}(f_3, f_4) \cos 2\pi (f_4 t_4 - f_3 t_3)\, df_3\, df_4 \tag{6.53}$$

$$= \int_0^{\infty}\int_{f_L - F}^{f_H - F} \delta(f_4 - f_3 - F) \cos 2\pi (f_4 t_4 - f_3 t_3)\, df_3\, df_4 \tag{6.54}$$

$$+ \int_0^{\infty}\int_{f_L + F}^{f_H + F} \delta(f_4 - f_3 + F) \cos 2\pi (f_4 t_4 - f_3 t_3)\, df_3\, df_4$$

Cases where $F > f_L$ are slightly more complicated and will not be considered.

See sketch below for first quadrant area



On reversing the order of integration in Eq. (6.54) and then integrating with respect to f_4, one obtains

$$R_{xz}(t_3, t_4) = \int_{f_L - F}^{f_H - F} \cos 2\pi\left[f_3(t_4 - t_3) + Ft_4\right] df_3 + \int_{f_L + F}^{f_H + F} \cos 2\pi\left[f_3(t_4 - t_3) - Ft_4\right] df_3 \tag{6.55}$$

It is a straightforward exercise to now determine $R_{xz}(t_3, t_4)$ for arbitrary t_3 and t_4. For the special point where $t = t_3 = t_4$, one obtains the cross-covariance term.

$$R_{xz}(t, t) = \int_{f_L - F}^{f_H - F} \cos 2\pi Ft \, df_3 + \int_{f_L + F}^{f_H + F} \cos 2\pi Ft \, df_3 = 2(f_H - f_L) \cos 2\pi Ft \tag{6.56}$$

when $F \leqq f_L$. This result is nonstationary since it is a function of t.

5. Joint Properties of $\{y(t)\}$ and $\{z(t)\}$

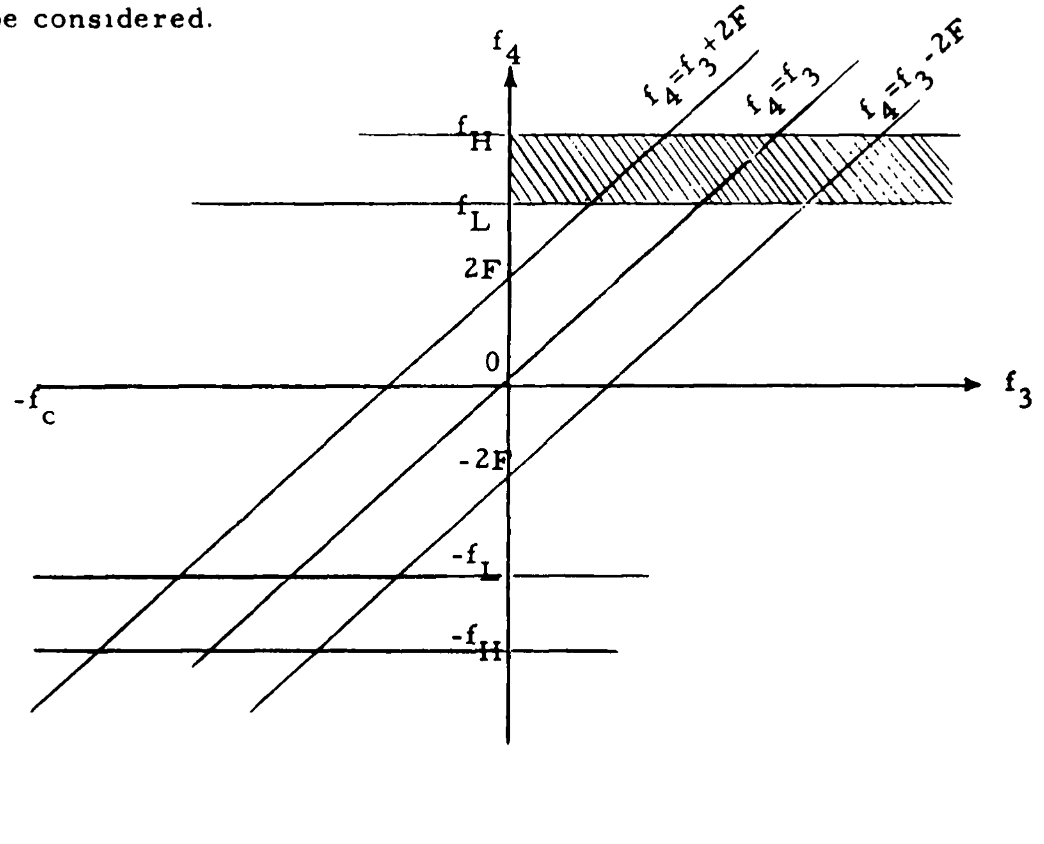
From Eq. (5. 70), the nonstationary cross-power spectral density function $S_{yx}(f_3, f_4)$ is given in this example by

$$S_{yz}(f_3, f_4) = \mathcal{H}(f_4)\, S_y(f_3, f_4) \tag{6.57}$$

where $\mathcal{H}(f_4)$ is defined by Eq. (6. 5), namely,

$$\begin{aligned} \mathcal{H}(f_4) &= 1 \quad ; \quad f_L \leq |f| \leq f_H \\ &= 0 \quad ; \quad \text{otherwise} \end{aligned} \tag{6.58}$$

and $S_y(f_3, f_4)$ is derived in Eq. (6. 26). It follows that $S_{yz}(f_3, f_4)$ exists only along the lines $f_4 = f_3$ and $f_4 = f_3 \pm 2F$ in the (f_3, f_4) plane where $\mathcal{H}(f_4) = 1$, and $S_x(f_4 \pm F) = 1$, $S_x(f_3 \pm F) = 1$, see Eq. (6. 26). The sketch below assumes that $2F \leq f_L$. For simplicity, other cases will not be considered.



It will be assumed that $f_c \geqq (f_H + F)$ so as to have $S_x(f_4 \pm F) = 1$ inside the region $f_4 < f_H$. From Eqs. (4. 16) and (6. 57), for $2F \leqq f_L$, the cross-correlation function $R_{yz}(t_3, t_4)$ becomes

$$R_{yz}(t_3, t_4) = 2 \int_0^{\infty} \int_0^{\infty} S_{yz}(f_3, f_4) \cos 2\pi (f_4 t_4 - f_3 t_3) \, df_3 \, df_4$$

$$= \frac{1}{2} \int_{f_L}^{f_H} \int_0^{\infty} \left[2\delta(f_4 - f_3) + \delta(f_4 - f_3 + 2F) + \delta(f_4 - f_3 - 2F) \right] \cos 2\pi (f_4 t_4 - f_3 t_3) \, df_3 \, df_4 \tag{6. 59}$$

$$= \int_{f_L}^{f_H} \left\{ \cos 2\pi f_4 (t_4 - t_3) + \frac{1}{2} \cos 2\pi \left[f_4 (t_4 - t_3) + 2Ft_3 \right] + \frac{1}{2} \cos 2\pi \left[f_4 (t_4 - t_3) - 2Ft_3 \right] \right\} df_4$$

$$= (1 + \cos 4\pi F t_3) \int_{f_L}^{f_H} \cos 2\pi f_4 (t_4 - t_3) \, df_4 \tag{6. 60}$$

which is integrated easily if desired.

Finally, the cross-covariance term is given by

$$R_{yz}(t, t) = (f_H - f_L)(1 + \cos 4\pi F t) \text{ when } 2F \leqq f_L \tag{6. 61}$$

This concludes the cases of the example that will be treated here. Further work is required to cover other possible parameter relations that might be considered. The results carried out here illustrate the application of formulas derived in preceding sections for analyzing input-output relations of nonstationary random processes through time-varying linear systems.

7. SAMPLING CONSIDERATIONS FOR FLIGHT VEHICLE VIBRATION PROBLEMS

7.1 INTRODUCTION

When measuring the vibration environment in a flight vehicle, the flight is usually much too long to record in its entirety. With available tape speeds of 30 or even 60 inches per second, often only a small portion of the vibration flight history can be recorded. It is easy to arrive at the conclusion that in order to obtain a representative picture of the vibration environment for the entire flight, samples should be taken during the various phases of the flight, each sample being only a few seconds long. However, it is more difficult to arrive at a quantitative conclusion as to how representative these samples are for this flight, and how accurately the vibration environment of another flight can be predicted.

A considerable amount of work has been done in developing quantitative results for procedures used in quality control and inspection sampling plans. Generally a sample is drawn from a population, measurements are made on the sample and sample parameters are calculated, which then provide an estimate of the corresponding population parameters. In addition, confidence limits can be established for the population parameters, providing a quantitative measure of the risk of having made a wrong estimate.

It would seem that such a procedure could also be established for measuring the vibration environment in flight vehicles. However, there are a number of difficulties that have to be resolved. For example, the vibration samples are not from a population of discrete items, but from a time-dependent function. Often this time dependent function is considered as random. Even the concept of "population" is not clear. It could be the vibration record for one complete flight or for several flights or maybe just for a particular flight phase. One very important requirement in inspection sampling is that the population parameters, which are to be estimated, do not change with time. Yet, for example, the rms acceleration level in a missile does vary considerably throughout the length of the flight. This requires the consideration of "nonstationarity" concepts.

Another question involves a choice of sampling at random or periodically. Then the "optimum" length of the sample has to be determined, taking into account the effect of sample length on the accuracy of the parameters to be estimated. It is the purpose of this section to look at the various aspects of "sampling of a time-dependent process" and to propose methods that will result in obtaining the maximum amount of information from a minimum amount of vibration data with a known confidence in the results.

The collection of vibration records under consideration may be, for example, different samples of the vibration amplitude response at a particular point on the structure of a given flight vehicle during a long mission. Other examples might be a collection of vibration records as obtained from many different locations on the structure of the given flight vehicle during one flight; or the collection of records might be obtained from the same location but over many different flights of the given flight vehicle. A number of different flight vehicles might also be involved. Clearly, there are a large number of possible physical variations that might be considered.

Regardless of the physical origin of the samples, the main problem is to analyze over-all statistical properties associated with the collection of samples. In some situations, prior analysis will have been made of statistical characteristics of each sample by itself.

A second problem, where it is applicable, is to determine in advance an appropriate number of samples which should be gathered for later analysis. A properly selected finite set of samples can reproduce the underlying population such that more samples provide little additional information. Attention to this matter can result in substantial savings both in gathering data and subsequent processing. This matter is discussed in Section 7.2 under the heading of Random Sampling, a review of material from Ref.[1]. In Section 7.3, periodic sampling of random data is shown to lead to equivalent results if data is both random and stationary.

From a collection of vibration records (samples), many parameters may be calculated for comparison with one another, such as mean square

values, power spectral density functions, amplitude probability density functions, extreme values, et cetera. Assume the records are obtained with instrumentation that has no DC response so that the mean value of each record is automatically zero. This agrees with much physically observed random data. For definiteness, assume for the present discussion that the individual mean square values are calculated. A single number, the mean square value, now represents each record. Each such number will have associated with it a statistical uncertainty, based on the number of degrees of freedom involved in its measurement. Two cases arise according to whether or not one has available and utilizes quantitative information of this uncertainty. Formulas for estimating uncertainty in measured sample values are discussed in Section 7.4, and methods for testing fundamental assumptions are discussed in Section 7.5.

The collection of sampled values can originate either from a single experiment (e.g., vibration data from a single point during one flight), or from a number of different experiments (e.g., vibration data from one or many points during one or many flights). For the sake of having a convenient summary of appropriate formulas in one place, Section 7.6 reviews material from Ref [1] for carrying out a statistical evaluation of data from single points (single experiments). This leads naturally to the next Section 8 of this report devoted to statistical procedures for evaluating data from many points (many experiments).

7.2 RANDOM SAMPLING

A major consideration in any data analysis is the extraction of the maximum amount of information from the minimum amount of data. Selection of an appropriate sampling scheme (random or periodic) can both decrease the volume of data required and still reproduce the population with desired precision. Knowing the mean time between samples, the distribution of these times, and the length of each sample, quantitative results can be obtained allowing the prediction of any given (vibration) level occurring, how often this level might occur, how well the samples represent the entire range of (vibration) levels, and the

minimum number of samples required for a given confidence in the results.

For reasons of simplicity both in instrumentation and in later statistical analysis, the sample length should be of fixed duration. In a random sampling scheme, only the time interval between samples should be random. Prior knowledge of the sample length, and of the frequency range being recorded, enables one to make valid statistical estimates about the accuracy of various measurements of interest, such as power spectrum measurements or probability density measurements. On the practical side, instrumentation problems of recording such data appear to be simplified if it is decided in advance that each sample length should be of a certain definite duration.

Figure 7.1 is a block diagram of a recommended procedure to follow in the selection of a random sampling plan. The blocks will be discussed in order and illustrated by examples. Ref. [1] may be consulted for more detailed comments.

Block A Experiment Specification and Equipment

Reference [1] gives detailed procedures to follow for specifying equipment for the flight vehicle vibration situation. Some similar analysis would be necessary in other physical problems.

Block B Sample Length Determination, Mean Time Between Samples

Several physical constraints exist such as, for example, the length of a sample should be at least as long as the period of the lowest

frequency of interest. Besides the physical constraints, engineering and statistical considerations are required to resolve interrelations existing between the sample length and the mean time between samples. These matters are reviewed in the text for Blocks C and D, and extended in considerable detail in Sections 7.4, 7.5, and 7.6.

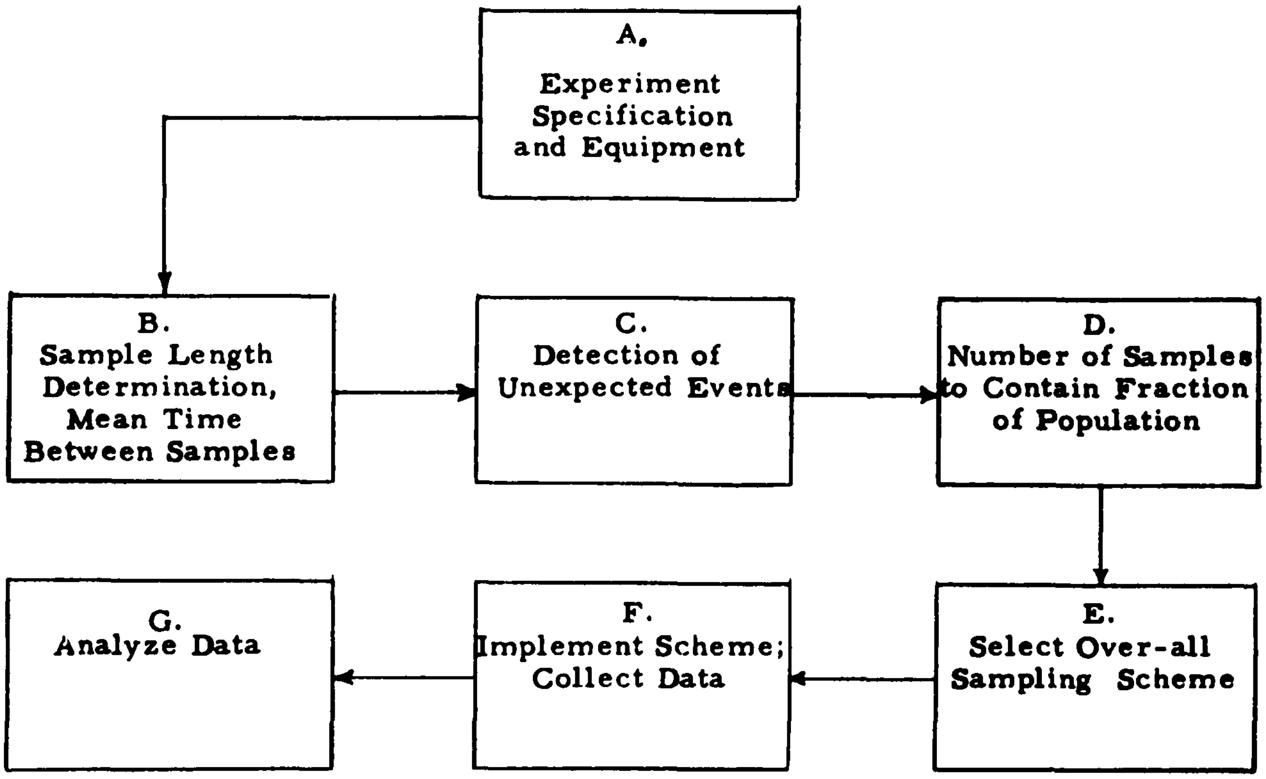


Figure 7.1. Over-all Recommended Procedure for Selection of Sampling Scheme.

Block C Detection of Unexpected Events

Certain important unexpected events might be missed in sampling from a random process. It is of interest to know the probabilities of missing these events for a given sample length and a given mean time between samples. Considered from the other direction, if a probability

is prescribed, then the mean time between samples, and the sample length will be partly determined. For the subsequent discussion, let

e = length of unexpected event in seconds

T = length of each sample in seconds

$\bar{L}$ = mean time between samples in seconds (measured from center to center)

Fig.7.2. shows a possible sequence, given that e will occur once during time $\bar{L}$.

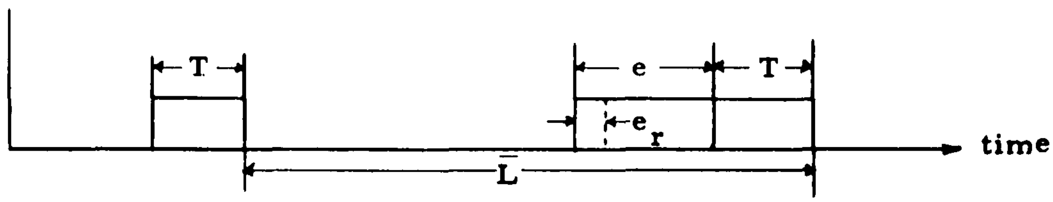


Figure 7.2. Example of Random Sampling

The average probability of missing all parts of e, denoted by $\bar{P}$, is

$$\bar{P}(\text{missing } e) = \frac{\bar{L} - (T + e)}{\bar{L}} = 1 - \frac{T + e}{\bar{L}} \quad (7.1)$$

assuming $e > 0$ and $(T + e) < \bar{L}$. This equation does not apply to situations of predetermined fixed sample lengths and fixed time between samples. For predetermined fixed sampling times,

$$P(\text{missing } e) = \begin{cases} 0 \text{ if } e \text{ occurs inside sample length} \\ 1 \text{ if } e \text{ occurs outside sample length} \end{cases}$$

The average probability of recording any part of e is

$$\bar{P}(\text{recording } e) = \frac{T + e}{\bar{L}} \quad (7.2)$$

For example, if $L = 100$ seconds, $T = 5$ seconds and $e = 10$ seconds

$$\bar{P}(\text{recording } e) = \frac{5 + 10}{100} = 0.15$$

This might appear to be quite low, but if e occurs only once, it might not be of too great interest. If e occurs, for example, 10 times during a 1000 second interval, a much higher probability would be desired and in fact does exist. Using the same values for e, T, and $\bar{L}$ the average recording probability is now

$$\bar{P}(\text{recording } e) = 1 - \left(1 - \frac{T + e}{L}\right)^{10} = 0.84$$

A more pertinent question than that of recording any part of e would be; "What is the average probability of recording at least 1 second of a 10 second event?" Let e_r be the minimum portion of e to be recorded. Then Eq.(7.2) becomes

$$\bar{P}(\text{recording at least } e_r) = \frac{T + (e - e_r)}{\bar{L}} \qquad (7.3)$$

If the probability density function for the time between samples is known, then a confidence interval can be established for the probability of missing e. Assuming, for instance, a normal (Gaussian) distribution for L with mean value $\bar{L}$ and with standard deviation σ, the probability of missing all of e is estimated by

$$\frac{(\bar{L} - \lambda\sigma) - (T + e)}{\bar{L} - \lambda\sigma} < P(\text{missing } e) < \frac{(\bar{L} + \lambda\sigma) - (T + e)}{\bar{L} + \lambda\sigma} \qquad (7.4)$$

where λ is the number of standard deviations required for a given confidence. The interval for the probability of recording any part of e is

$$\frac{T + e}{\bar{L} + \lambda\sigma} < P(\text{recording } e) < \frac{T + e}{\bar{L} - \lambda\sigma} \qquad (7.5)$$

and for recording at least a portion e_r of e is

$$\frac{T + (e - e_r)}{\bar{L} + \lambda\sigma} < P(\text{recording at least } e_r) < \frac{T + (e - e_r)}{\bar{L} - \lambda\sigma} \qquad (7.6)$$

For example, if L is normally distributed with $\bar{L} = 100$ sec, $\sigma = 5$ sec, $T = 5$ sec, $e = 10$ sec, $e_r = 1$ sec, and $\lambda = 2$, it can be said with 95% confidence that the probability of recording at least 1 second of the event e falls in the interval

$$\frac{5 + (10 - 1)}{100 + 10} < P(\text{recording at least } e_r) < \frac{5 + (10 - 1)}{100 - 10}$$

$$0.127 < P < 0.156$$

The above equations can assist in determining a desirable sample length and mean time between samples. Of course, judgment is necessary here in determining how long an unexpected event should be to be considered important, and what is the permissible miss probability. Reference [1] contains further discussion and examples.

Block D Number of Samples to Contain Fraction of Population

Another criteria to help determine the sample length and mean time between samples is the number of samples required to predict the total (vibration) life history from these samples, namely the total range of possible events.

If one is concerned about deviations in only one direction, (i.e., below the sample maximum value), the proportion of the population occurring below the maximum sample value can be calculated with a known confidence from the equation, see Reference A,

$$(P)^N = \alpha \qquad (7.7)$$

where

$$P = \text{proportion of population}$$
$$1 - \alpha = \text{confidence coefficient}$$
$$N = \text{sample size}$$

If deviations in both directions are of concern, the proportion of the population occurring within the maximum and minimum sample values is given by

$$NP^{N-1} - (N - 1) P^N = \alpha \tag{7.8}$$

where N, P and α are defined as above.

Figure 7.3 gives a plot of Eq.(7.7) for $P = 0.85$, 0.90, and 0.95. This curve clearly demonstrates, that for $P = 0.90$, very little additional confidence would be gained by taking a continuous record, no matter how long, rather than approximately 50 samples. An assumption made here is that the population is infinite, which is generally considered valid if the population is at least ten times as large as the number of samples taken. Given values for P and α, N may be obtained from Figure 7.3. For example, if $P = 0.95$ and $(1 - \alpha) = 0.90$, then $N = 45$. Conversely, if $N = 45$ and $\alpha = 0.10$, then $P = 0.95$.

It may be of interest to know the probability of a sample value x occurring between $\mu \pm \lambda\sigma$ where μ is the mean value, σ is the standard deviation of the population, and λ is a constant. Making no assumptions about the population distribution, the Tchebycheff inequality may be applied. This gives

$$\text{Prob}\left[\mu - \lambda\sigma < x < \mu + \lambda\sigma\right] > 1 - \frac{1}{\lambda^2} \tag{7.9}$$

If a normal (Gaussian) distribution exists, the following relation holds which yields a higher probability value, namely,

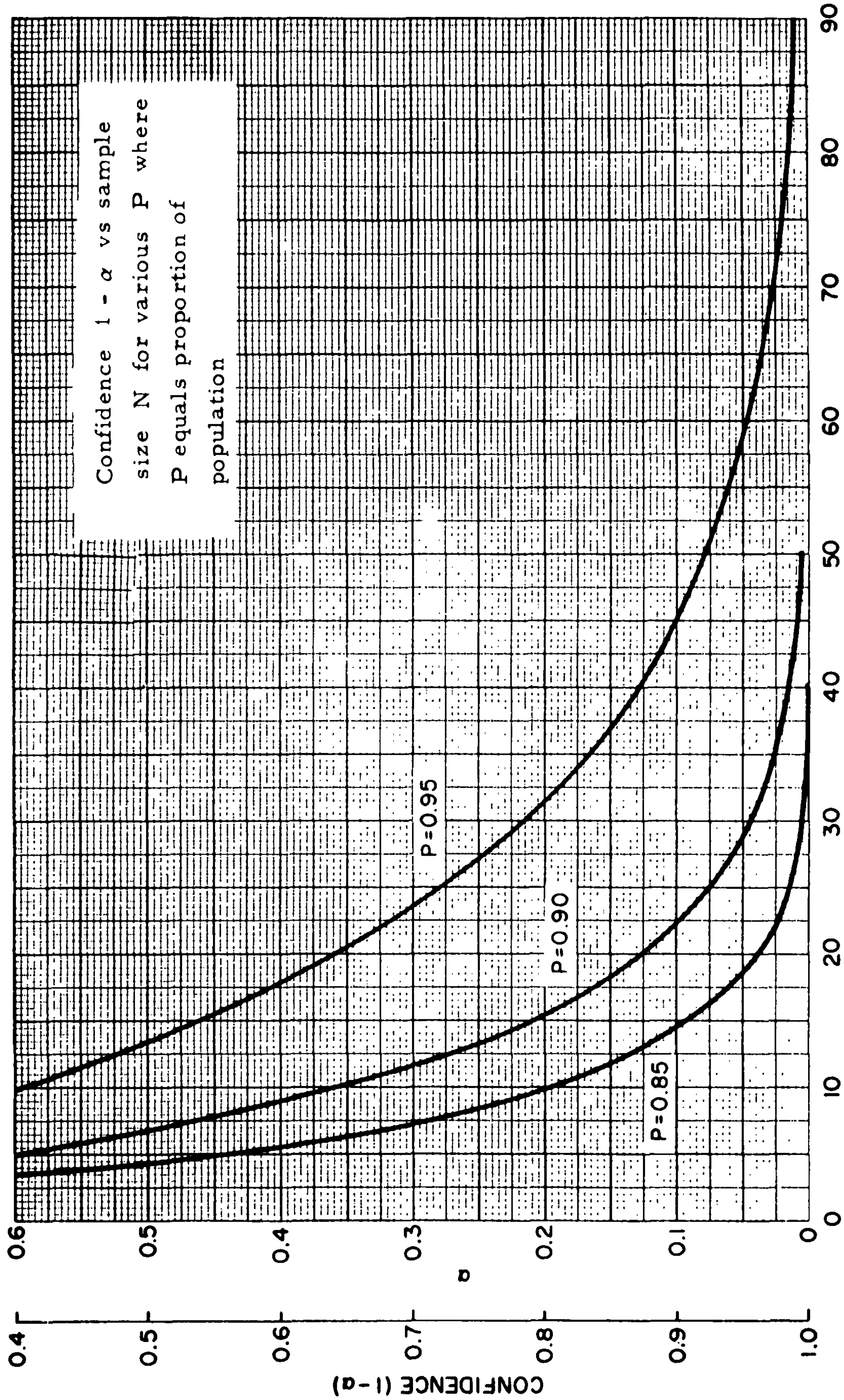


Figure 7.3. Plot of Equation $P^N = \alpha$

$$\text{Prob}\left[\mu - \lambda\sigma < x < \mu + \lambda\sigma\right] = \int_{\mu-\lambda\sigma}^{\mu+\lambda\sigma} \frac{1}{\sigma\sqrt{2\pi}} \exp\left[\frac{-(x-\mu)^2}{2\sigma^2}\right] dx \quad (7.10)$$

This distribution is well tabulated and widely used.

Table 7.1 below gives a comparison of probability results about the sample values based upon the Tchebycheff inequality and the assumption of normality.

Table 7.1

λ	1	2	3
Normal	0.6827	0.9545	0.9973
Tchebycheff	0	0.7500	0.8889

$\text{Prob}\left[\mu - \lambda\sigma < x < \mu + \lambda\sigma\right]$

Block E Select Over-all Sampling Scheme

After the above steps have been accomplished, an over-all sampling scheme may now be devised. This again will require some judgment in adjusting the sample length, mean time between samples, and number of samples to fit together as a meaningful whole. See Reference [1] and Section 8 here for detailed examples for the flight vehicle vibration situation.

Block F Implement Scheme; Collect Data

Again one is referred to Reference [1] for a discussion of many of the problems, such as instrumentation, which are involved in final implementation of a random sampling scheme.

Block G Analyze Data

At this point one begins the analyses presented in Reference [1] and in other parts of this report.

7.3 PERIODIC SAMPLING OF RANDOM DATA

If data is assumed or verified to be truly random, as opposed to being periodic or even containing some periodicities, then one can choose a fixed time $\bar{L} = L$ between samples. When L is in seconds, the period of the samplings is (1/L) samples per second. Now, Eqs. (7.4), (7.5), and (7.6) have $\sigma = 0$, and Eqs. (7.2) and (7.3) with $\bar{L} = L$ state the probability of recording unexpected events for periodic sampling. The validity of this procedure is dependent on the data being random. If the data has periodicities, then periodic sampling can confuse the results through aliasing effects wherein high frequencies appear as low frequencies.

With periodic sampling, rather than random sampling, it is easier to detect underlying linear trends in the data such as may occur, for example, in ballistic missile flights. Once detected and analyzed, these trends can be removed so that further analysis can be devoted to the other data characteristics. The most important of these is the question of stationarity of the data.

If the data is both random and stationary, then the data can be collected at arbitrary random starting times, and sampled periodically thereafter, without introducing any errors. However, if the data is nonstationary, then results are dependent upon the particular times of observation and should be analyzed by special methods as discussed in Sections 2 through 6 of this report. For nonstationary data, one should impose a fixed starting time requirement and express the results as a function of this starting time. Whether sampling randomly or periodically, Eqs. (7.7) and (7.8) which predict the range of expected events are valid only when the data is stationary.

For these reasons and others, it is clear that it is important to verify that the data is both random and stationary. Where possible, Table 7.1 shows that it is desirable also to test for normality so as to improve the statistical confidence about the range of sample values.

7.4 UNCERTAINTIES IN ESTIMATED PARAMETERS

This section and the next section deal with the problem of coordinating all of the various requirements that must be satisfied in the determination of the length T of each of the samples. The reader will note that sample length requirements vary widely depending on which parameter one desires to estimate, even if one choses the same confidence limits throughout. The primary purpose in designing any experiment is to obtain the "maximum" amount of information from a "minimum" amount of data. The "maximum" amount of information is defined here as that required to satisfy a predetermined confidence in the results. The "minimum" amount of data is interpreted as the shortest sample, and the least number of samples, needed to meet the predetermined confidence, resulting in minimum cost and manpower expended.

First, a brief review will be presented of the various formulas between sample length and the associated statistical errors for each of certain desired basic parameters to be estimated. This is followed by discussion of tests for fundamental assumptions, and then by a summary of the sample length relationships.

7.4.1 Lowest Frequency of Interest

In order to make sure that the lowest frequency of interest can be detected, at least one full cycle of this frequency has to be recorded. Based on engineering judgment it is recommended that at least two cycles be recorded. This will avoid possible loss of this frequency, in case minor variations in sample length occur. In equation form

$$T = \frac{2}{f_{min}} \tag{7.11}$$

where f_{min} is the minimum frequency of interest.

7.4.2 Mean Square Estimates

When estimating, for example, the mean-square acceleration levels during a flight, the confidence in the measurements can be determined from knowledge of the equivalent number of statistical degrees of freedom n

$$n = 2BT_1 \qquad T_1 \leq T \tag{7.12}$$

where B is the bandwidth in terms of an idealized rectangular filter, and T_1 is the integrating time (or averaging time) of an ideal integrator. It should be noted here that unless the filter has an attenuation of at least 60 db per octave, an equivalent noise bandwidth B_n has to be computed for B in Eq. (7.12). See Ref. [1, p. 4-84] and Section 12.3.2 of this report.

Equation (7.12) shows also that the maximum number of degrees of freedom for a sample of length T occurs when the integration time T_1 is equal to T. This in turn would then result in the maximum statistical confidence that can be obtained. In other words, if the integration time T_1 is less than the sample length T, the number of degrees of freedom is determined from T_1. If the integration time is equal to or greater than the sample length T, the number of degrees of freedom is limited by the sample length T. Using an averaging time that is longer than the sample will therefore do nothing to improve the statistical accuracy of the data, but will only result in a waste of time and effort.

Now consider a signal with a Gaussian distribution and a true mean square acceleration of σ^2. From a mean square measurement of s^2 from a sample of length T, a confidence interval for the true mean square value can be determined based on a knowledge of the number of degrees of freedom n, Ref. [1, p. 7-10] and Section 13 of this report.

A limited table of confidence intervals for a mean square measurement of 1.0 as a function of the degrees of freedom is presented in Table 7.2.

No. of Degrees of Freedom n		2	10	20	40	60	120
80% Confidence Interval	Lower Limit	0.43	0.62	0.70	0.77	0.81	0.85
	Upper Limit	9.48	2.05	1.67	1.37	1.29	1.19
95% Confidence Interval	Lower Limit	0.27	0.49	0.58	0.67	0.72	0.79
	Upper Limit	39.20	3.07	2.08	1.63	1.48	1.31

Table 7.2. Confidence Intervals for Mean Square Measurements

To illustrate Table 7.2, assume that a sample is two seconds long ($T=2$) and is analyzed using an ideal filter with a 30 cps bandwidth, and with a true integrator having an integration time T_1 of two seconds. If this results in a mean square acceleration reading of 10 g^2, one can be 95% confident (see Table 7.2) that the true mean square acceleration in the aircraft during this two-second time period was between 7.9 and 13.1 g^2, since $n = 2BT_1 = 2\,(30)(2) = 120$ (or that the true rms acceleration was approximately between 2.81 and 3.62 g's). It should be noted that in order to obtain the overall mean square value of a sample, the integration time T_1 of the instrument has to be exactly equal to the sample length T.

Often RC filters are used to perform the integration. It has been shown in Ref. [1, p. 7-12] that this results in a different number of degrees of freedom, namely

$$n = 4BK \tag{7.13}$$

where $K = RC$ is the time constant of the integrator. Now, since the output voltage of an RC integrator will reach the input voltage after

approximately 4 time constants have elapsed, the record should be at least 4 times as long as the time constant of the integrator. For a two-second sample, this would require a time constant of 0.5 seconds.

Often the vibration signal is recorded on magnetic tape which makes it possible to form a continuous loop for each sample. In this case the time constant of the RC integrator is limited only by Eq. (7.13). This equation limits the maximum number of degrees of freedom obtainable from a record length T . Combining this with Eq. (7.12) one finds

$$K = \frac{1}{2} T \qquad (7.14)$$

as the optimum value of K. Increasing the value of K beyond that indicated by Eq. (7.14) will do nothing to improve the statistical quality of the measurements.

The quality of the estimate can also be expressed in terms of the "standard error" ϵ or normalized variance ϵ^2 of the measurement as follows:

$$\epsilon^2 = \frac{1}{BT_1} \; ; \quad T_1 \leq T \qquad (7.15)$$

This equation is useful for confidence interval estimation if it can be assumed that the measurements are normally distributed. Also, it is often used by itself as a measure of the error in the results. The assumption of normality is a good approximation if the number of degrees of freedom exceeds 120. Then the square root of $\epsilon = \sqrt{1/BT_1}$ is equal to one standard deviation of the measurement and a 95% confidence interval is given by $s^2 \pm 2\epsilon$. For example, if B = 60 cps and T_1 = T = 5 seconds, then n = 600 and $\epsilon = \sqrt{1/300} = .058$. For a measurement of 10 g^2 the 95% confidence interval is then approximately 9.88 to 10.12 g^2.

7.4.3 Power Spectral Density Estimates

The basic measurement performed for power spectral density analysis is the mean square value determination. Therefore the same arguments apply here as those developed in Section 7.4.2 above. For example, assume a sample record is six seconds long and the bandwidth of the analyzer filter from resolution considerations is determined as 10 cps, resulting in $n = 120$ degrees of freedom. The maximum statistical quality attainable in the power spectrum measurement has now been fixed by Eq. (7.12). Analysis techniques employed from this point on cannot increase the statistical accuracy of the measurement beyond that defined by the 120 degrees of freedom. However, the statistical accuracy can be further degraded if the analyzer filter scans the frequency range at too fast a rate. This problem will now be discussed for constant bandwidth power spectra estimates and for constant percentage power spectra estimates.

(a) Constant Bandwidth Power Spectra Estimates

From Ref. [1, p. 7-29], the maximum scan rate, (S.R.), below which the statistical accuracy will not be affected has been established as

$$\text{S.R.} \leq \frac{B}{T_1}; \quad T_1 < T \tag{7.16}$$

Another problem that must be considered is the transient response of the analyzer filter. On the basis of proper filter response the scan rate should be limited to

$$\text{S.R.} \leq \frac{B^2}{8} \tag{7.17}$$

If an RC integrator is employed, the scan rate should be limited to

$$\text{S.R.} \leq \frac{B}{4K} \tag{7.18}$$

(b) Constant Percentage Power Spectra Estimates

For constant percentage analysis, the analyzer filter bandwidth increases in direct proportion to the center frequency of the filter. Here $B = Pf_c$ where P is the resolution (some constant fraction less than one) and f_c is the center frequency. For constant percentage analysis, Eq. (7.12), (7.16), (7.17), and (7.18) become respectively

$$n = 2\, Pf_c\, T_1 \tag{7.19}$$

$$S.R. \leq \frac{Pf_c}{T_1} \tag{7.20}$$

$$S.R. \leq \frac{(Pf_c)^2}{8} \tag{7.21}$$

$$S.R. \leq \frac{Pf_c}{4K} \tag{7.22}$$

7.4.4 Probability Density Estimates

The uncertainty for probability density estimates is expressed in terms of a normalized standard error ϵ. Experimental results yield the following expression (see Section 14 of this report) for a probability density estimate $\hat{p}(x)$

$$\epsilon \approx \left[\frac{0.07}{\Delta x\, \hat{p}(x)\, \overline{\nu_0}\; T_1}\right]^{1/2} ; \; T_1 < T \tag{7.23}$$

The quantity Δx is the amplitude window width, $\overline{\nu_0}$ is the expected number of zero crossings per unit time, and T_1 is the averaging time. (This is usually equal to the record length T). It is assumed here that the bandwidth B is not narrow with $B << \overline{\nu_0}$, see Section 14. For analog instruments which take time averages by smoothing with a low pass RC filter, $T_1 = 2K$ (for $T_1 < T$), where K is the filter RC time constant.

The value for $\overline{\nu_0}$ is a function of the signal's frequency characteristics. For signals having a uniform power spectrum between two effective cutoff frequencies f_a and f_b, the term $\overline{\nu_0}$ is given by

$$\overline{\nu_0} = 2\left[\frac{f_a^{\,2} + f_a f_b + f_b^{\,2}}{3}\right]^{1/2} \tag{7.24}$$

If the low frequency cutoff is at $f_a = 0$ the term $\overline{\nu_0}$ is given by

$$\overline{\nu_0} = 1.15\, f_b \tag{7.25}$$

In applying Eq. (7.23) for cases where the signal is a representative record from a stationary random process, confidence intervals are found by using $\left[1 \pm \lambda\epsilon\right]\hat{p}(x)$ where λ is a normal deviate (for $n > 120$). For example, $\lambda = 1$ gives a 68% confidence interval, and $\lambda = 2$ gives a 95% confidence interval. For $n < 120$, instead of λ, the t statistic should be used.

Solving Eq. (7.23) for T_1, gives the equation for the required averaging time for any desired standard error ϵ. This also becomes the expression for the record length T if the averaging time is equal to the record length, namely

$$T = T_1 = \frac{\epsilon^2\, \Delta x\, \hat{p}(x)\, \overline{\nu_0}}{0.07} \tag{7.26}$$

Hypothetical cases based on Eq. (7.26) appear in Section 14 of this report. These cases show the record lengths which are required for typical vibration and acoustics problems. For example, from Table 14.12 in Section 14, one sees that a record length of about 16 seconds would be required to obtain estimates with $\epsilon = 0.10$ out to $\pm$ 3 rms, assuming $\overline{\nu_0} = 1000$, $\Delta x = 0.1$, and $\hat{p}(x)$ follows a Gaussian distribution with an rms value of 1.0.

So far it has been assumed that the averaging time T_1 is equal to the record length T. In reality, sample records are often available,

which are longer than the maximum averaging time of the analyzing instrument.

If the analyzer averages by linear integration, a series of estimates will be obtained over a single record length T. If these estimates are then averaged, the normalized variance of this average will be equal to ϵ^2/N, where N is the number of estimates obtained from a single record T, each of length $T_1 = (T/N)$.

If the analyzer averages with a low pass RC filter, the effective averaging time T_1 is equal to 2K (for $T_1 \ll T$), where K is the time constant of the RC filter.

Many probability density analyzers estimate p(x) by continuously sweeping the amplitude window Δx. Here the Sweep Rate (S. R.) must be slow enough to permit a given amplitude level to be viewed by Δx over the entire averaging period, that is

$$S.R. \leq \frac{\Delta x}{T_1} \tag{7.27}$$

If this condition is not met, the uncertainty of the estimates will be increased. Specifically, if S. R. is greater than $\Delta x/T_1$, the effective record length will be

$$T_{eff} = \frac{\Delta x}{S.R.} \tag{7.28}$$

Here, if an RC averager is employed, the scan rate should be limited to

$$S.R. \leq \frac{\Delta x}{4K} \tag{7.29}$$

7.4.5 Correlation Function Estimates

For correlation function estimates, the normalized mean square error is also a function of the ideal bandwidth B and the record length T. The exact expression for this mean square error is quite complicated and

is also a function of the signal to noise ratio. In the vicinity of R(0), approximately up to the first zero crossing, the mean square error ϵ^2 is

$$\epsilon^2 \approx \frac{1}{BT} \tag{7.30}$$

At R(0), ϵ^2 is a conservative estimate of the time error of the correlation function estimate and this error becomes progressively worse as the time difference τ increases.

If τ is varied continuously an additional error is introduced. This is given by Ref. [1, p. 7-68] for large BT (say BT>100) as

$$\epsilon^2 < \lambda_1^2 \, BT \tag{7.31}$$

where $\lambda_1 = \Delta\tau/T$ and $\Delta\tau$ is the smallest increment in τ to be distinguished.

7.5 TESTS OF FUNDAMENTAL ASSUMPTIONS

Statistical tests for randomness and for stationarity are investigated experimentally in Sections 15 and 16 of this report. Appropriate theoretical material is included with the experimental results. One of the tests for randomness and one of the tests for stationarity are based on measurement uncertainties in sample mean square values. These tests are as follows.

7.5.1 Test for Randomness

Consider a sample record of length T obtained from a vibration response x(t) with an equivalent ideal bandwidth B. Assume x(t) is stationary over the sample record length T (to be verified by other procedures discussed in Section 16), and that x(t) has an approximately Gaussian amplitude probability density function with a true mean square value of ms_x. If the vibration response is actually

random, a mean square measured value $\overline{x^2}$ from the sample record will have a sampling distribution with a variance given in normalized terms as follows.

$$\epsilon_o^2 = \frac{\mathrm{Var}\left[\overline{x^2}\right]}{ms_x^2} \approx \frac{1}{BT_a} ; \quad T_a \leq T \tag{7.32}$$

where T_a = averaging time. For RC averaging with time constant K, one sets $T_a = 2K$ when $2K < T$, and $T_a = T$ when $2K > T$.

If a periodic component is present in the otherwise random vibration response, the normalized variance for a mean square measurement becomes

$$\epsilon^2 = \frac{\mathrm{Var}\left[\overline{x^2}\right]}{ms_x^2} = \frac{\left(2\frac{p}{r} + 1\right)}{\left(\frac{p}{r} + 1\right)^2} \epsilon_o^2 \tag{7.33}$$

where

p = mean square value for periodic portion

r = mean square value for random portion

$ms_x = ms_x(\text{periodic}) + ms_x(\text{random}) = p + r$

From Eq. (7.33), the variance of a mean square measurement will be reduced as the level of the periodicity is increased relative to the random background (as p/r increases). This relationship lays the groundwork for a statistical hypothesis test for randomness.

Assume a collection of N independent mean square values, $\overline{x_i^2}$ $(i = 1, 2, 3, \ldots, N)$, are measured from a stationary vibration response. This collection may be obtained from a single sample record by averaging over each of N equally long segments with an averaging time $T_a = (T/N)$. The variance of the mean square measurements can be estimated from the sample variance s^2 for the measured values $\overline{x_i^2}$ as follows.

$$s^2 = \frac{1}{N} \sum_{i=1}^{N} \left(\overline{x_i^2} - \overline{x^2} \right)^2 \tag{7.34}$$

$$\overline{x^2} = \frac{1}{N} \sum_{i=1}^{N} \overline{x_i^2}$$

Here, $\overline{x^2}$ is the mean square value average over the collection length $T = N T_a$. The expected values for the sample variance s^2 and the sample mean $\overline{x^2}$ are as follows.

$$E\left[s^2\right] = \left(\frac{N-1}{N}\right) \mathrm{Var}\left(\overline{x^2}\right) \tag{7.35}$$

$$E\left[\overline{x^2}\right] = ms_x$$

Then, the normalized variance for a mean square measurement may be estimated by

$$\epsilon^2 = \frac{s^2}{\left(\overline{x^2}\right)^2} \tag{7.36}$$

In Eq. (7.36), both s^2 and $\overline{x^2}$ are random variables. However, it is shown in Section 13 that the variability of $\overline{x^2}$ is negligible compared to the variability of s^2 if the quantity $2BT > 40N$ or $2BT_a > 40$. If this requirement is met, it may be assumed that $\overline{x^2} = ms_x$ for the problem at hand.

The sample variance s^2 from Eq. (7.36) will have a distribution associated with the chi-squared distribution as follows.

$$\frac{s^2}{\mathrm{Var}\left(\overline{x^2}\right)} \sim \frac{\chi^2(N-1)}{N} \tag{7.37}$$

where "$\sim$" means "distributed as", and $\chi^2_{(N-1)}$ is a chi-squared distribution with (N-1) degrees of freedom. From the relationships in Eqs. (7.33) and (7.36), it follows that

$$\frac{\hat{\epsilon}^2}{\epsilon^2} \sim \frac{\chi^2_{(N-1)}}{N} \quad ; \quad 2BT_a > 40 \tag{7.38}$$

From Eq. (7.38), the following probability statement may be made.

$$\text{Prob}\left[\frac{\hat{\epsilon}^2}{\epsilon^2} \geq \frac{\chi^2_{(N-1);(1-\alpha)}}{N}\right] = (1-\alpha) \tag{7.39}$$

where α is the level of significance, and $(1-\alpha)$ is the confidence level.

Let it be hypothesized that a sampled vibration response is random. If this is true, $\epsilon^2 = \epsilon_o^2$, and the hypothesis H_0 is

$$H_0 : \hat{\epsilon}^2 = \epsilon_o^2 \tag{7.40}$$

If a periodic component is present, $\hat{\epsilon}^2$ will be equivalent to some ϵ^2 less than ϵ_o^2 as defined in Eq. (7.33), which is why a one-sided probability statement is used in Eq. (7.39). Then, H_0 may be tested by computing $\hat{\epsilon}^2$ from a collection of mean square measurements, computing ϵ_o^2 from the BT_a product for the measurements, and comparing the ratio of $\hat{\epsilon}^2/\epsilon_o^2$ to the χ^2 limit in Eq. (7.39) at any desired level of significance α. The region of acceptance for H_0 is

$$\frac{\hat{\epsilon}^2}{\epsilon_o^2} \geq \frac{\chi^2_{(N-1);(1-\alpha)}}{N} \tag{7.41}$$

If $\hat{\epsilon}^2/\epsilon_o^2$ is greater than the above noted limit, H_0 is accepted and the vibration response is considered random. If $\hat{\epsilon}^2/\epsilon_o^2$ is less than the noted limit, H_0 is rejected and there is reason to suspect that a periodic component is present in the vibration response. The appropriate values in Eq. (7.41) for any desired level of significance can be obtained from most statistics books.

Further discussion of this test for randomness, and other tests, appears in Section 15.

7.5.2 Test for Stationarity

Assume a collection of N independent mean square values, $\overline{x_i^2}$, $(i = 1, 2, 3, \ldots, N)$, are measured from a random vibration response. This collection may be obtained from a single sample record by averaging over each of N equally long segments with an averaging time $T_a = (T/N)$. If the vibration response is stationary with a mean square value of ms_x, each measurement $\overline{x_i^2}$ will be an estimate of ms_x and, thus, will be equivalent to one another. One procedure for testing the collection of measurements for equivalence is as follows.

This procedure is identical in concept to the statistical test for randomness described earlier. If the vibration response is random and stationary, the mean square measurement $\overline{x_i^2}$ will have a sampling distribution with a normalized variance of ϵ_o^2, as defined in Eq. (7.32). For a collection of N mean square measurements, the actual normalized variance ϵ^2 may be estimated by $\hat{\epsilon}^2$, as defined in Eqs. (7.34) and (7.36). The ratio of $\hat{\epsilon}^2/\epsilon^2$ will be distributed as shown in Eq. (7.38).

The following probability statement may be made.

$$\text{Prob}\left[\frac{\hat{\epsilon}^2}{\epsilon^2} \leq \frac{\chi^2_{(N-1);\alpha}}{N}\right] = (1-\alpha) \qquad (7.42)$$

Let it be hypothesized that a sampled random vibration response is stationary. If this is true, $\epsilon^2 = \epsilon_o^2$, and the hypothesis H_0 is

$$H_0 : \hat{\epsilon}^2 = \epsilon_o^2 \qquad (7.43)$$

If the vibration response is nonstationary, $\hat{\epsilon}^2$ will be equivalent to some undefined ϵ^2 greater than ϵ_o^2, which is why a one-sided (upper tail) probability statement is used in Eq. (7.42). Then, H_0 may be tested by computing $\hat{\epsilon}^2$ from a collection of mean square measurements, computing ϵ_o^2 from the BT_a product for the measurements, and comparing the ratio of $\hat{\epsilon}^2/\epsilon_o^2$ to the χ^2 limit in Eq. (7.42) at any desired

level of significance α. The region of acceptance for H_0 is

$$\frac{\hat{\epsilon}^2}{\epsilon_o^2} \leq \frac{\chi^2_{(N-1);\alpha}}{N} \tag{7.44}$$

If $\hat{\epsilon}^2/\epsilon_o^2$ is less than the above noted limit H_0 is accepted and the vibration response is considered stationary. If $\hat{\epsilon}^2/\epsilon_o^2$ is greater than the noted limit, H_0 is rejected and there is reason to suspect that the vibration response is nonstationary.

Further discussion of this test for stationarity, and other tests, appears in Section 16.

It is clear that the above test can be combined with the test for randomness discussed in Section 15, by using a two-sided χ^2 test. If it is hypothesized that the vibration response is random and stationary, the region of acceptance for the double hypothesis H_0 is

$$\frac{\chi^2_{(N-1);(1-\alpha/2)}}{N} \leq \frac{\hat{\epsilon}^2}{\epsilon_o^2} \leq \frac{\chi^2_{(N-1);\alpha/2}}{N} \tag{7.45}$$

In Eq. (7.45), since $(\alpha/2)$ replaces α, the lower limit is less than the limit for the randomness test alone, as given in Eq. (7.41). Similarly, the upper limit is higher than the limit for the stationary test alone, as given in Eq. (7.44). Thus, a test of the double hypothesis will involve a wider region of acceptance for a given confidence level $(1 - \alpha)$. To maintain a desired Type II error, more measurements are required for the combined test than for each individual test.

7.5.3 Test for Normality

Section 17 of this report contains results from an experimental study which develops a test for normality based on expected measurement uncertainties in amplitude probability density estimates.

7.6 SAMPLE LENGTH DETERMINATION

From the results of Section 7.4 and 7.5, one can see that there are five basic parameters and three tests of fundamental assumptions that involve the sample length. These are listed below in order of increasing sample length usually required for a given mean square error or confidence in the results.

Parameters:

1) Lowest Frequency of Interest
2) Mean Square Value
3) Correlation Function
4) Power Spectral Density
5) Amplitude Probability Density

Tests of Assumptions:

1) Randomness
2) Stationarity
3) Normality

Qualitative reasons for stating that the necessary sample length T increases in going from parameter (1) to parameter (5) are as follows. The lowest frequency of interest gives a bound for T which is not dependent upon a measurement error ϵ. However, in order to reduce a mean square value measurement error, one must increase T appropriately. Uncertainties in mean square measurements and correlation function measurements have the same error formula for the maximum point of the correlation function at zero delay. This is not true for non-zero correlation delays. For these correlation points, T must be increased to maintain the same error. Next, bandwidths used for power spectral density function measurements are generally smaller than for mean square value measurements since one desires to resolve sharp spectral peaks. To maintain a given error, these bandwidth reductions must be compensated for by longer record lengths. Finally, amplitude probability density estimates require the longest sample lengths because one desires to measure these probability density functions out to values of extreme amplitudes.

In order to arrive at the minimum sample length, based on some predetermined maximum allowable error, the following steps should be considered.

Step 1. Since the estimation of the vibration environment is of primary importance, the sample length should always be based on a minimum acceptable mean square error for any of the four statistical parameters desired. (Note that f_{min} is not a "statistical" parameter.) Using this sample length, one can then determine the confidence in the tests of fundamental assumptions.

Step 2. All instrumentation that will be used in the estimation of the various parameters should have an ideal averaging time equal to or greater than the sample length determined from the appropriate equations. If the ideal averaging time is less than the sample length, the error will increase and the confidence will decrease in the results. If RC averaging is employed, the minimum time constant K = RC should satisfy

$$K \geq \frac{1}{2} T \tag{7.14}$$

Step 3. No matter which parameters are to be estimated, Eq. (7.11) should be used as a check to determine if the requirement for the lowest frequency of interest can be met, namely,

$$T \geq \frac{2}{f_{min}} \tag{7.11}$$

For example, if $f_{min} = 2$ cps, then T should be ≥ 1 second.

Step 4. If the amplitude probability density is one of the parameters to be estimated, the sample length should be determined from Eq (7.26). It will usually result in the longest sample for any given mean square error as compared to the other parameters.

$$T = \frac{\epsilon^2 \Delta x \hat{p}(x) \overline{\nu}_0}{0.07} \tag{7.26}$$

If the amplitude window Δx is swept continuously, the maximum allowable sweep rate that can be used, without increasing the error decided upon should be calculated from the appropriate Eqs. (7.27) or (7.29).

Step 5. If the amplitude probability density is not required, the power spectral density will now usually govern the sample length which should be determined from Eq. (7.15). By solving for T, one obtains

$$T = \frac{1}{B\epsilon^2} \tag{7.46}$$

In order not to increase the error decided upon, the maximum allowable sweep rates should be calculated from the appropriate Eqs. (7.16), (7.17), or (7.18).

Step 6. If only correlation functions and/or mean square values are desired, Eq. (7.46) can again be used for sample length determination.

For correlation function estimates, one also has to consider the rate of change of τ. The maximum allowable sweep rate should be calculated from the appropriate Eq. (7.31).

7.7 STATISTICAL EVALUATION OF DATA FROM SINGLE EXPERIMENT

When the sample length has been decided upon, one has to determine the effect of the number of samples on the over-all confidence of the experiment. The "experiment" might be the measurement of vibration levels at a particular point during a certain flight phase or during the whole flight.

By <u>sample size</u> will be meant here the number, N, of samples to be obtained in any one experiment. The term <u>number of experiments</u> will denote the number k of repetitions of a particular experiment (e.g., the number of flights) in each of which N samples are taken. Altogether, there would be Nk samples. It is important to select N and k, if possible, so that one can make good statistical predictions both with data from within a single experiment, as well as with data from experiment to experiment. If N and k are merely given values, and not at one's choosing, then it is important to be able to calculate the statistical accuracy to be attached to predictions based on the given values.

Estimates of a true population mean μ and a true population variance σ^2 as calculated from a sample of size N will be denoted by $\bar{x}$ and s^2, respectively. The estimates $\bar{x}$ and s^2 will be random variables, differing for each sample of size N, while the true values μ and σ^2 are constants. The standard deviation σ is, of course, the positive square root of the variance.

The discussion in this section is devoted to the case of a single experiment for which $k = 1$. Cases of multiple experiments (e.g., simultaneous data from many points or from many flights) are analyzed in Section 8.

7.7.1 Case 1: No consideration of uncertainty in measured sample values.

Suppose N independent parameter estimates (e.g., mean square values), denoted by x_j, $j = 1, 2, \ldots, N$, are obtained in a single experiment, and suppose the statistical uncertainly in each sample value is unknown or ignored. The first problem is to obtain estimates of the sample mean and sample variance to be associated with the N parameter values from this single experiment.

By definition, the sample mean is obtained by calculating

$$\bar{x} = \frac{\sum_{j=1}^{N} x_j}{N} \tag{7.47}$$

and the sample variance, that is, the mean square value about the mean value, by

$$s^2 = \frac{\sum_{j=1}^{N} (x_j - \bar{x})^2}{N} \tag{7.48}$$

One can now show that the expected value of $\bar{x}$ is

$$E(\bar{x}) = \mu \tag{7.49}$$

and the expected value of s^2 is

$$E(s^2) = \left(\frac{N - 1}{N}\right) \sigma^2 \tag{7.50}$$

where μ and σ are the true population mean and variance, respectively.

A block diagram for analyzing a set of parameter values from a single experiment is drawn in Figure 7.4. The various confidence intervals and tolerance intervals in Fig. 7.4 can be determined in parallel as shown.

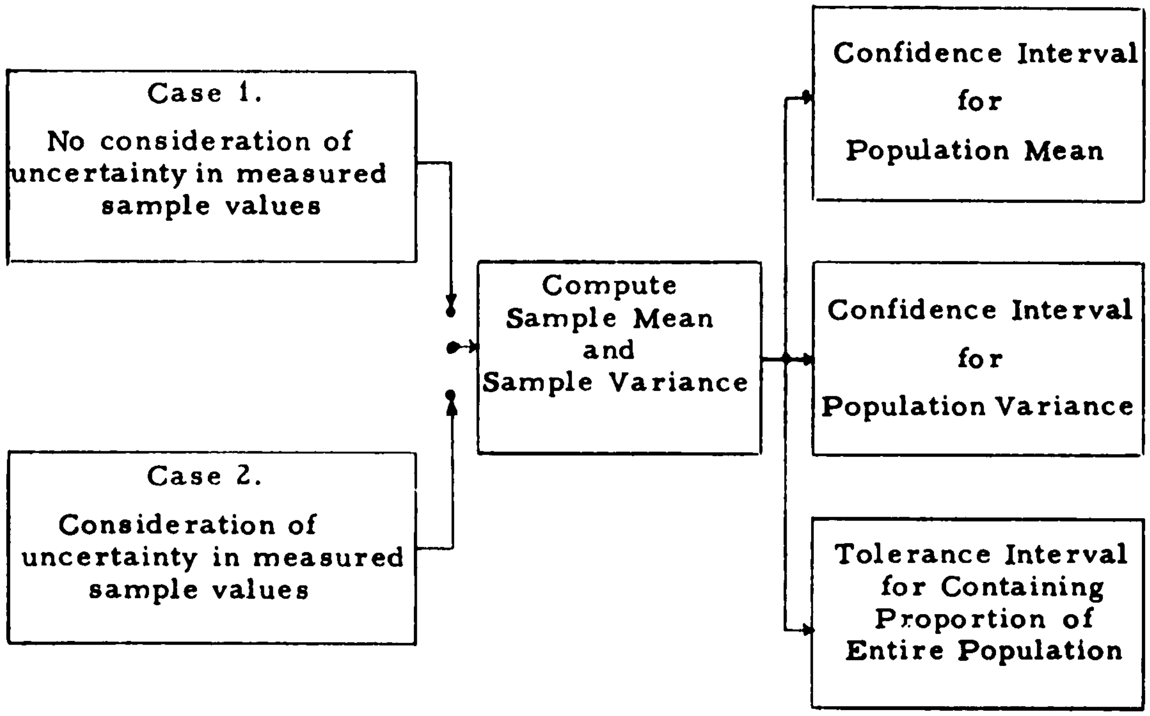


Figure 7.4. Recommended Procedure for Analyzing a Set of Values from a Single Experiment

Confidence Interval for Population Mean

If the N parameter values are assumed to originate from an underlying normally distributed population (a hypothesis which can be tested by statistical methods discussed in Reference [1]. then a confidence interval can be formed around the sample mean value $\bar{x}$ so as to contain the true population mean value μ within any desired level of significance α. The interpretation of confidence is that if the experiment of estimating $\bar{x}$ and s from N sample values is repeated many times,

then the probability is $(1 - \alpha)$ that the true mean μ will lie in the interval bounded by $\bar{x} \pm t_{(\alpha/2)}\ s/\sqrt{N-1}$, that is

$$\text{Prob}\left[\bar{x} - t_{(\alpha/2)}\ \frac{s}{\sqrt{N-1}} \leq \mu \leq \bar{x} + t_{(\alpha/2)}\ \frac{s}{\sqrt{N-1}}\right] = (1 - \alpha) \tag{7.51}$$

where α is some desired proportion $(0 \leq \alpha \leq 1)$ and $t_{(\alpha/2)}$ is taken from the tables of the t distribution with $(N - 1)$ degrees of freedom at the $(\alpha/2)$ percent value.

For example, assume $\alpha = .05$ and a sample of size $N = 6$ with $\bar{x} = 12.3$ and $s = 2.19$. In the tables of the t-distribution one finds for 5 degrees of freedom $t_{2.5} = 2.57$. Using Eq. (7.51),

$$\text{Prob}\left[12.3 - (2.57)\ \frac{2.19}{\sqrt{5}} \leq \mu \leq 12.3 + (2.57)\ \frac{2.19}{\sqrt{5}}\right] = .95$$

$$= \text{Prob}\left[9.8 \leq \mu \leq 14.8\right] = .95$$

Confidence Interval for Population Variance

A confidence interval may also be computed around the sample variance s^2 so as to contain the true population variance σ^2 within any desired level of significance α. This is given by

$$\text{Prob}\left[\frac{Ns^2}{\chi^2_{(\alpha/2)}} \leq \sigma^2 \leq \frac{Ns^2}{\chi^2_{1-(\alpha/2)}}\right] = (1 - \alpha) \tag{7.52}$$

where χ_p^2 is taken from tables of the χ^2 distribution with $(N - 1)$ degrees of freedom at the p percent value. Above, $p = (\alpha/2)$ and $1 - (\alpha/2)$.

As an example, assume the same sample values as above. Then from Eq. (7.52),

$$\text{Prob}\left[\frac{6(4.80)}{12.8} \leq \sigma^2 \leq \frac{6(4.80)}{0.4}\right] = .95$$

$$= \text{Prob}\left[2.2 \leq \sigma^2 \leq 67.0\right] = .95$$

where $\chi^2_{2.5} = 12.8$ and $\chi^2_{97.5} = 0.4$ are found from a table of χ^2 with 5 degrees of freedom.

Large Sample Size

For large values of N, say $N \geq 30$, confidence intervals for μ and σ can be approximated closely using tables of the normal distribution. The results for large N are

$$\text{Prob}\left[\bar{x} - \lambda_\alpha \frac{s}{\sqrt{N}} \leq \mu \leq \bar{x} + \lambda_\alpha \frac{s}{\sqrt{N}}\right] = (1 - \alpha) \qquad (7.53)$$

$$\text{Prob}\left[s - \lambda_\alpha \frac{s}{\sqrt{2N}} \leq \sigma \leq s + \lambda_\alpha \frac{s}{\sqrt{2N}}\right] = (1 - \alpha) \qquad (7.54)$$

where λ_α is the α-percent value of the normal distribution, as defined by considering deviations in both directions from the mean value. The previous percent values for the t and χ^2 distributions considered deviations only in one direction.

Assume $\bar{x} = 12.3$ and $s = 2.19$ but with a sample size $N = 36$. Then for $\alpha = .05$, $\lambda_{.05} = 1.96$ is found from a table of the normal distribution. Using Eq. (7.53)

$$\text{Prob}\left[12.3 - (1.96)\frac{2.19}{\sqrt{36}} \leq \mu \leq 12.3 + (1.96)\frac{2.19}{\sqrt{36}}\right] = .95$$

$$= \text{Prob}\left[11.6 \leq \mu \leq 13.0\right] = .95$$

and from Eq. (7.54),

$$\text{Prob}\left[2.19 - (1.96)\frac{2.19}{6\sqrt{2}} \leq \sigma \leq 2.19 + (1.96)\frac{2.19}{6\sqrt{2}}\right] = .95$$

$$= \text{Prob}\left[1.7 \leq \sigma \leq 2.7\right] = .95$$

<u>Tolerance Intervals</u>

A <u>tolerance interval</u> around a sample mean value $\bar{x}$, calculated from a sample of size N, is defined as an interval that will contain at least a proportion $(1 - \alpha)$ of the underlying population a desired percentage P of the time. This percentage P is called a confidence coefficient. The tolerance interval is bounded by the values

$$\bar{x} \pm Ks \tag{7.55}$$

where K is a numerical coefficient depending upon N, α, and P, which is tabulated in Ref. [1].

For example, assuming an underlying normal distribution, for a sample of size 100, the tolerance interval that will contain at least 90% of the population with 95% confidence is given by

$$\bar{x} \pm 1.87\, s$$

Considered another way, the above example states that the probability is 0.95 that at least 90% of all population values will lie in the interval shown.

If the true population mean, μ, and true standard deviation, σ, were known precisely, then regardless of the size of N, there is 100% confidence that at least a proportion $(1 - \alpha)$ of the population lies inside the interval

$$\mu \pm \lambda_\alpha \sigma \tag{7.56}$$

where λ_α is the α-percent value of the normal distribution. For example, 100% confidence exists that at least 95.5% of the population will lie inside the interval $\mu \pm 2\sigma$. On the other hand, for $K = 2.0$, only 75% confidence exists that at least 90% of the population lies inside the interval $\bar{x} \pm 2s$, if $\bar{x}$ and s are calculated from 15 samples.

For large N, where μ and σ are approximated closely by $\bar{x}$ and s, the coefficient K approaches the normal α-percent value λ_α and the confidence coefficient P approaches 100%. In the previous example, where $K = 2.0$, there is 90% confidence that at least 90% of the population, based on calculating $\bar{x}$ and s from 33 samples, lie inside the interval $\bar{x} \pm 2s$; there is 95% confidence that at least 90% of the population using 50 samples lie inside this same interval and 99% confidence for at least 90% of the population using 90 samples.

The main point of this discussion is that, for a proportion $(1 - \alpha) = 0.90$ and a constant $K = 2.0$, little increased confidence (i.e., higher P) is gained by choosing N larger than, say, $N = 50$. In the example above, changing N from 50 to 90 increased the confidence only from 95% to 99% that at least 90% of the population would fall in the interval $\bar{x} \pm 2s$.

7.7.2 Case 2. Consideration of uncertainty in measured sample values.

Suppose N independent parameter values, denoted by x_j, $j = 1, 2, \ldots, N$ are obtained in a single experiment. Suppose also that the variance associated with each of these values is estimated to be s_j^2, where s_j^2 are assumed to be the same from experiment to experiment. None of the s_j^2 are permitted to be zero. How should the sample mean and sample variance be calculated now for the N parameter values of the experiment?

For this situation, define the sample mean by

$$x = \sum_{j=1}^{N} a_j x_j \tag{7.57}$$

where

$$a_j = \frac{1}{s_j^2 \sum_{j=1}^{N} (1/s_j^2)} \tag{7.58}$$

Define the sample variance for the N values by

$$s^2 = N \sum_{j=1}^{N} a_j^2 s_j^2 = \frac{N}{\sum_{j=1}^{N} (1/s_j^2)} \qquad (7.59)$$

Since s_j^2 is an estimate of the true variance σ_j^2, the coefficients

$$a_j \approx \frac{1}{\sigma_j^2 \sum_{j=1}^{N} (1/\sigma_j^2)} \qquad (7.60)$$

To allow Eq.(7.60) to replace Eq.(7.58) would require prior information of the true variances σ_j^2, usually not known.

In view of the assumption that s_j^2 is a fixed estimate of σ_j^2 from experiment to experiment, the coefficients a_j are constants from experiment to experiment. Now, from the fact that $\sum_{j=1}^{N} a_j = 1$, it follows that the expected value of the sample mean is

$$E(\bar{x}) = \sum_{j=1}^{N} a_j E(x_j) = \mu \sum_{j=1}^{N} a_j = \mu \qquad (7.61)$$

and the expected value of the sample variance is

$$E(s^2) = N \sum_{j=1}^{N} a_j^2 E(x_j^2) = N \sum_{j=1}^{N} a_j^2 \sigma_j^2 \qquad (7.62)$$

If s_j^2 equals the same value for all j, then $a_j = (1/N)$ for all j. For this special situation,

$$\bar{x} = \frac{1}{N} \sum_{j=1}^{N} x_j \qquad \text{with} \qquad E(\bar{x}) = \mu \qquad (7.63)$$

$$s^2 = \frac{1}{N} \sum_{j=1}^{N} s_j^2 \qquad \text{with} \qquad E(s^2) = \sigma^2 \qquad (7.64)$$

when $\sigma^2 = E(s_j^2)$ is assumed to be the same for all j.

The remainder of the analysis for Case 2 now proceeds as in Case 1, with $\bar{x}$ and s^2 defined by Eqs. (7.57) and (7.59) instead of by Eqs. (7.47) and (7.48). For example, if $N = 2$, with $x_1 = 16$, $x_2 = 20$, and $s_1^2 = 1$, $s_2^2 = 3$, one calculates $a_1 = (3/4)$, $a_2 = (1/4)$. Then $\bar{x} = 17$ and $s^2 = 1.5$.

7.8 STATISTICAL EVALUATION OF DATA FROM MANY EXPERIMENTS

The most useful statistical analysis for evaluating a collection of experiments comes under the general heading of "analysis of variance." The object of this analysis is to test the hypothesis that the mean values of some particular parameter from experiment to experiment are not significantly different. If this hypothesis can be verified, the results of the several experiments may be combined. Then, the resulting large sample size may be used for more precise estimates of the parameters of interest. One has thus simultaneously verified an important assumption and has a large amount of data for parameter estimation

For instance, in the case of vibration in a flight vehicle, one may take a large amount of data from a single flight and make precise estimates of parameters for that single flight. However, it is obviously of value to be able to use the results to make predictions to other flights conducted under similar conditions. Clearly, one is on somewhat tenuous grounds making predictions to other flights no matter how large the sample from the single flight. However, when some consistency from flight to flight has been verified, predictions may be made to other similar flights with much greater confidence.

These matters are discussed in the next Section 8 of this report.

7.9 REFERENCES

1. Bendat, J. S., Enochson, L. D., Klein, G. H., and A. G. Piersol. The Application of Statistics to the Flight Vehicle Vibration Problem, ASD TR 61-123, Aeronautical Systems Division, Air Force Systems Command, USAF, Wright-Patterson AFB, Ohio. December 1961. ASTIA AD 271 913.

8. ANALYSIS OF VARIANCE PROCEDURES FOR EVALUATING VIBRATION DATA FROM MANY POINTS

8.1 INTRODUCTION

In the prediction of the over-all vibration life history of flight vehicles, analysis of variance techniques can be employed to determine if significant differences between vibration measurements exist, and, if they do exist, the magnitude of the variance of these measurements can be estimated.

The general hypothetical problem is as follows. Samples, each of the same length, are taken at random for the entire flight of a given flight vehicle at several points within each flight vehicle. These measurements are then repeated for several flights of the same flight vehicle or for other flight vehicles of the same general class (i.e., B-52's, F-105's, etc.). Before the complete set of vibration measurements can be pooled to predict the over-all vibration environment, it has to be determined if these measurements are statistically equivalent. Specifically, the vibration engineer has to know: (1) are all the measurements within each flight at one particular point equivalent; (2) are the measurements between the various points within each flight vehicle equivalent, and (3) are the measurements from flight to flight equivalent.

If the measurements within each flight are not equivalent, the data can be said to be "nonstationary." One solution would then be to break up each flight into various flight phases, within which the measurements are equivalent, i.e., within which the environment is "stationary." If the measurements between the various points are not equivalent, a separate prediction will have to be made for each of the points.

Finally, if the measurements from flight to flight are not statistically equivalent, the data cannot be pooled, and an over-all prediction cannot be made. The vibration engineer will then have to look for the physical cause of these differences and may have to design a new experiment. For example, assume that four flights were made, and one particular flight shows a considerably higher vibration environment than the other three and an analysis of variance indicates a significant difference between flights. If the vibration engineer decides that the higher vibration levels were due to bad weather during the one flight, possibly the new experiment should consist of more flights, several of which should be made during the bad weather, so that a more representative sample is obtained.

It is realized that time and money may make it difficult to repeat such a program and, therefore, the engineer should do his best to obtain a representative sampling on the first go-around. The analysis of variance cannot help the engineer in many aspects of the experimental design. However, it can be of considerable assistance in indicating proper methods of data collection such that the maximum information may be obtained from the minimum amount of data.

Another possible use for the analysis of variance techniques is to determine if a single vibration specification could be written for a general type of aircraft (e. g. ,all fighters, or all bombers, etc.). If data is available for a number of different aircraft, all of one type, this data can then be tested to determine if it is statistically equivalent.

The following sections will describe the simpler analysis of variance techniques that can be employed to arrive at the appropriate decisions. The primary portion of this discussion has been taken from a book by A. H. Bowker and G. J. Lieberman entitled "Engineering Statistics," published by Prentice Hall in May 1961. For additional information, the reader is referred to this text, Ref [2].

The subsequent discussion for the one-way analysis of variance is for a large part a repetition of a portion of Section 5. 4, "Statistical Results from Repeated Experiments", of Ref. [1]. The subject is directly extended here to the more complicated two-way analysis of various procedures. Also, the two different interpretations of the analysis of variance procedures are presented here which were ignored in Ref. [1] in the interest of simplicity. The discussion in Ref. [1] corresponds to the random-effects model (model II) one-way analysis described here.

The repetition of material in Ref. [1] is given in the interest of completeness, and also for the introduction of different notation and computing procedures which are highly useful in extending to two-way (and higher) analyses. At certain points in this section, the notation will be reconciled with that of Ref. [2] in order that one may cross reference with a minimum of confusion.

The material here does not tell one how to go about interpreting the data if the analysis indicates pooling may be allowed. However, this subject is discussed in some detail in Ref. [1] and the reader may refer to that report for extensions in that direction.

8.2 ONE-WAY ANALYSIS OF VARIANCE

Before any predictions can be made, the vibration engineer has to determine whether or not the various flights are statistically equivalent. For this purpose alone, a one-way analysis of variance may be used. Since measurements may be taken at more than one location within each flight vehicle, this procedure should be applied separately to each of the points. For the measurements at one point only, if one desires to make a decision about the within flight and between flight measurements simultaneously, then the two-way analysis of variance procedure discussed in Section 8.3 should be used.

8.2.1 Mathematical Models

For one particular point on the structure, let x_{ij} be the jth measurement for the ith flight. It should be noted that x represents one sample. It could be the mean square acceleration for one sample, or the rms acceleration, or represent velocity, displacement, stress, or any other unit of measurement.

(a) Fixed Effects Model

This mathematical model is written as

$$x_{ij} = \zeta_i + \epsilon_{ij} \tag{8.1}$$

where ζ_i is the fixed effect due to the ith flight and ϵ_{ij} is a random effect assumed to be independently normally distributed with mean zero and variance σ^2.

ζ_i can be further subdivided into $\zeta_. + \phi_i$ where $\zeta_.$ is a component common to all flights and ϕ_i is a fixed effect peculiar only to ith flight. The dot (.) in place of a subscript indicates averaging over that subscript. This subdivision is accomplished by defining $\zeta_.$ as the average of all the ζ_i, i.e.,

$$\zeta_. = \frac{\sum_{i=1}^{r} \zeta_i}{r} \tag{8.2}$$

and defining ϕ_i as the deviation of the ζ_i from the average, i.e.,

$$\phi_i = \zeta_i - \zeta_. \tag{8.3}$$

Thus, ϕ_i has the property $\sum_{i=1}^{r} \phi_i = 0$. The x_{ij} can then be written as

$$x_{ij} = \zeta_. + \phi_i + \epsilon_{ij} \tag{8.4}$$

where x_{ij} is assumed to be normally distributed with mean $\zeta_. + \phi_i$ and variance σ^2.

The purpose here is to draw some conclusions about whether the flights are equivalent. This is the same as making statements about the equality of the ϕ_i. If the ϕ_i are equal, they all must be equal to zero. This model is called the "fixed effects" model because the ϕ_i are fixed effects and conclusions derived from the experiment will apply <u>only</u> to the flights made. Conclusions about other flights cannot be made with the interpretation afforded by this model.

(b) <u>Random Effects Model</u>

If it is desired to extend the conclusions to other flights (which is usually the case) a "random effects" model has to be used. Here the experimenter has to make sure that the flights during which he takes measurements are chosen at random from all possible flights.

The mathematical model is still written as

$$x_{ij} = \zeta_. + \phi_i + \epsilon_{ij} \tag{8.5}$$

but the interpretation of the ϕ_i is different. Here ϕ_i is a random variable having a probability distribution which is the distribution of flight effects. The ϵ_{ij}, $i = 1, \ldots, c$, are assumed to be normally distributed with zero mean and common variance σ^2. (A test for this assumption is given in Ref. [1].) If the experiment were to be repeated, a new random sample would be drawn and different flights might be included.

In the fixed effects model, repeating the experiment would mean using the same four flights. Therefore, for the random effects model, all the flights are equivalent if $\sigma_{\phi}^{2} = 0$. This model is called the "random effects" model because ϕ is a random variable, and the conclusions derived from the experiment will be extended to all flights from which the random sample was drawn. In this model the x_{ij} are normally distributed with mean ζ and variance $\sigma^2 + \sigma_{\phi}^{2} = \sigma^2$ since σ_{ϕ}^{2} is hypothesized to be zero.

8.2.2 Computational Procedure

Assume that the following data has been taken for one particular location (see Table 8.1).

Flights		Measurements				
		1	2	3	j	c
	1	x_{11}	x_{12}	x_{13}	x_{1j}	x_{1c}
	2	x_{21}	x_{22}	x_{23}	x_{2j}	x_{2c}
	i	x_{i1}	x_{i2}	x_{i3}	x_{ij}	x_{ic}
	r	x_{r1}	x_{r2}	x_{r3}	x_{rj}	x_{rc}

Table 8.1 Arrangement of Vibration Data

An analysis of variance table, such as that shown in Table 8.2, will now have to be completed. This is done by following the computational procedure described below.

(1) Calculate totals for each flight

$$R_1, R_2, \ldots, R_r$$

(2) Calculate over-all total

$$T = R_1 + R_2 + \dots + R_r$$

(3) Compute crude total sum of squares

$$\sum_{i=1}^{r} \sum_{j=1}^{c} x_{ij}^2 = x_{11}^2 + x_{12}^2 + \dots + x_{rc}^2$$

(4) Calculate crude sum of squares between flights

$$\frac{\sum_{i=1}^{r} R_i^2}{c} = \frac{\left(R_1^2 + R_2^2 + \dots + R_r^2\right)}{c}$$

(5) Calculate correction factor due to mean

$$C.F. = \frac{T^2}{rc}$$

Then,

(6) $$SS_3 = (4) - (5) = \sum_{i=1}^{r} \frac{R_i^2}{c} - \frac{T^2}{rc}$$

(7) $$SS = (3) - (5) = \sum_{i=1}^{r} \sum_{j=1}^{c} x_{ij}^2 - \frac{T^2}{rc}$$

(8) $$SS_2 = (7) - (6) = SS - SS_3$$

The mean square values SS^*, SS_2^*, and SS_3^* can now be calculated by dividing by the appropriate degrees of freedom as shown in Table 8.2.

In the above computational procedure SS_3^* and SS_2^* correspond to the between group variance, $N s_{\bar{x}}^2$, and the within group variance, s^2, of Ref. [1] with c and r replacing N and k respectively. The major difference is that SS_3^* and SS_2^* are the unbiased estimates while $N s_{\bar{x}}^2$ and s^2 are biased. That is,

$$SS_3^* = \frac{k}{k-1} N s_{\bar{x}}^2 \qquad \text{and} \qquad SS_2^* = \frac{Nk}{k(N-1)} s^2$$

Source	Sum of Squares	Degrees of Freedom d. f.	Mean Square	Expected Mean Square for the Fixed Effects Model	Expected Mean Square for the Random Effects Model
Between Flights	$SS_3 = c \sum_{i=1}^{r} (\bar{x}_{i\cdot} - \bar{x}_{\cdot\cdot})^2$	$r - 1$	$SS_3^* = \frac{SS_3}{r-1}$	$\sigma^2 + \frac{\sum_{i=1}^{r} (\phi_i)^2}{r-1}$	$\sigma^2 + c\sigma_\phi^2$
Within Flights	$SS_2 = \sum_{i=1}^{r} \sum_{j=1}^{c} (x_{ij} - \bar{x}_{i\cdot})^2$	$r(c - 1)$	$SS_2^* = \frac{SS_2}{r(c-1)}$	σ^2	σ^2
Total	$SS = \sum_{i=1}^{r} \sum_{j=1}^{c} (x_{ij} - \bar{x}_{\cdot\cdot})^2$	$rc - 1$	$SS^* = \frac{SS}{rc-1}$	---	---

In this table $\bar{x}_{i\cdot} = \sum_{j=1}^{c} \frac{x_{ij}}{c}$; $\bar{x}_{\cdot\cdot} = \sum_{i=1}^{r} \sum_{j=1}^{c} \frac{x_{ij}}{rc}$

Table 8. 2 Analysis of Variance Table, One-Way Classification (Ref. [2], p. 297)

The F test is

$$F = \frac{SS_3^*}{SS_2^*} = \frac{\frac{k}{k-1}\; N s_{\bar{x}}^2}{\frac{Nk}{Nk - k}\; s^2} = \frac{s_{\bar{x}}^2 (Nk - k)}{s^2 (k - 1)}$$

which shows that the tests here and in Ref. [1] are equivalent. Other equivalences are easily seen. For instance, the so called "correction factor" (using the terminology of Ref. [2]) is the term $(\Sigma x)^2/Nk$ in the formula

$$s^2 = \frac{\Sigma x^2 - \frac{(\Sigma x)^2}{Nk}}{Nk}$$

for computing a sample variance.

(a) Decision Procedure, Fixed Effects Model

The hypothesis that all the flights are equal can be written as

$$H : \phi_1 = \phi_2 = \ldots = \phi_r \tag{8.6}$$

In fact, if the hypothesis is true, each ϕ_i must equal zero. It can be shown (Ref. [2], p. 295) that the hypothesis (8.6) is true, if the ratio SS_3^*/SS_2^* (see Table 8.2) is distributed as an F random variable with (r - 1) and r(c - 1) degrees of freedom. Therefore, the hypothesis (8.6) is accepted if

$$F = \frac{SS_3^*}{SS_2^*} \leq F_{\alpha;\, r-1,\, r(c-1)} \tag{8.7}$$

where α is the level of significance and $F_{\alpha; r-1, r(c-1)}$ is the upper α percentage point of the F distribution.

If the hypothesis (8.6) is rejected, it is possible to make statements about which flights differ. Tukey and Scheffé have developed a method which is described in Reference [2], page 295, where the average vibration level of each flight can be compared. A factor k is calculated as follows.

$$k = k^* \sqrt{\frac{SS_2^*}{c}} \tag{8.8}$$

where k^* is obtained from the table in Appendix 8.-A.

The appropriate degrees of freedom are those corresponding to the degrees of freedom of the within flight source in the analysis of variance table, i. e., $r(c - 1)$. Then any two flight averages differing by more than k, are said to differ significantly. The average vibration level of the ith flight is, of course, defined as

$$\bar{x}_{i\cdot} = \sum_{j=1}^{c} x_{ij}/c$$

(b) Decision Procedure, Random Effects Model

Testing the hypothesis about the homogeneity of flights is now equivalent to testing the hypothesis

$$H : \sigma_{\phi}^{2} = 0 \tag{8.9}$$

Just as in the fixed effects model, $F = SS_3^* / SS_2^*$ is distributed as an F random variable with $(r - 1)$ and $r(c - 1)$ degrees of freedom. Therefore, the hypothesis that $\sigma_{\phi}^{2} = 0$ is accepted if

$$\frac{SS_3^*}{SS_2^*} \leq F_{\alpha;r-1,\,r(c-1)} \tag{8.10}$$

In order to determine a measure of how much the flights differ, the quantity σ_{ϕ}^{2} can be estimated. An estimate of σ_{ϕ}^{2} is given by (Ref. [2], p. 307)

$$\sigma_{\phi}^{2} = \frac{SS_3^* - SS_2^*}{c} \tag{8.11}$$

If this quantity is negative, the estimate is taken to be zero. The probability of Type II error, β, i. e., the probability of accepting the hypothesis $\phi_i = 0$ or $\sigma_{\phi}^{2} = 0$ when in fact $\phi_i \neq 0$ or $\sigma_{\phi}^{2} \neq 0$, can also be estimated from a set of Operating Characteristic Curves. However, it is beyond the scope of this report to do so and the reader is referred to Ref. [2].

8.2.3 Numerical Example

Assume that from a series of flights made by a particular type of aircraft, four flights were selected at random and during these four flights, vibration measurements were taken at several points. These measurements were made at random during the take-off, cruise, and landing phases of the flight consisting of samples, each two seconds long. For each of the two-second samples, an over-all rms acceleration level was determined, resulting in the following data for one particular point:

Flight 1: 7, 8, 9, 8, 8, 5, 2, 1.5, 2.5, 2, 4, 4.5, 5, 4.5, 4
Flight 2: 8, 7, 9, 7, 8, 6, 2.5, 2, 1.5, 2, 4, 5, 4, 4.5, 4.5
Flight 3: 8, 9, 7, 9, 7, 5, 2.5, 2.5, 1.5, 2, 3.5, 4, 4.5, 4, 5
Flight 4: 7, 9, 8, 7, 9, 5, 1.5, 2, 2, 2.5, 4, 4, 3.5, 5, 3.5

Arranging this data in tabular form as that shown in Table 8.1, results shown in Table 8.3 are obtained.

Flights	Vibration Levels															
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	Totals
1	7	8	9	8	8	5	2.	1.5	2.5	2.	4.	4.5	5.	4.5	4.	75.0
2	8	7	9	7	8	6	2.5	2.	1.5	2.	4.	5.	4.	4.5	4.5	75.0
3	8	9	7	9	7	5	2.5	2.5	1.5	2.	3.5	4.	4.5	4.	5.	74.5
4	7	9	8	7	9	5	1.5	2.	2	2.5	4	4.	3.5	5.	3.5	73.0
											Grand Total					297.5

Table 8.3 Pairing of Flights and Vibration Levels

Following the computational procedure of Section 2.2 yields

(1) $R_1, R_2, R_3, R_4 = 75.0,\ 75.0,\ 74.5,\ 73.0$

(2) $T = 297.5$

(3) $$\sum_{i=1}^{4} \sum_{j=1}^{15} x_{ij}^2 = 1,822.25$$

(4) $$\sum_{i=1}^{4} \frac{R_i^2}{c} = \frac{22,129.25}{15} = 1,475.28$$

(5) $$\frac{T^2}{rc} = \frac{88,506.26}{60} = 1,475.10$$

(6) $$SS_3 = \sum_{i=1}^{r} \frac{R_i^2}{c} - \frac{T^2}{rc} = 0.18$$

(7) $$SS = \sum_{i=1}^{r} \sum_{j=1}^{c} x_{ij}^2 - \frac{T^2}{rc} = 347.15$$

(8) $$SS_2 = SS - SS_3 = 346.97$$

The completed analysis of variance table is shown in Table 8.4.

Source	Sum of Squares	Degrees of Freedom	Mean Square
Between Flights	$SS_3 = 0.18$	$r - 1 = 3$	$SS_3^* = \frac{SS_3}{3} = 0.06$
Within Flights	$SS_2 = 346.97$	$r(c-1) = 56$	$SS_2^* = \frac{SS_2}{56} = 6.20$
Total	$SS = 347.15$	$rc - 1 = 59$	$SS^* = \frac{SS}{59} = 5.88$

Table 8.4 Analysis of Variance Table, One-Way Classification

(a) Random Effects Model

Since the four flights were selected at random, and conclusions are to be drawn about all other similar flights, the random effects model should be used.

In Section 8. 2. 2(b), it was shown that testing the hypothesis about row effects (between flights) is equivalent to testing $H: \sigma_\phi^2 = 0$ and is accepted if

$$F = \frac{SS_3^*}{SS_2^*} \leq F_{\alpha;3,56}$$

from Eq. (8 10). Choosing a level of significance $\alpha = 0.05$ and looking up $F_{0.05;3,56}$ in the F tables (Ref. [2], p. 560) one obtains

$$F = \frac{SS_3^*}{SS_2^*} = \frac{0.06}{6.20} \leq F_{\alpha;3,56} = 2.76$$

and, therefore, accept $H: \sigma_\phi^2 = 0$.

An estimate of σ_ϕ^2 was given by Eq. (8. 11) and is

$$\hat{\sigma}_\phi^2 = \frac{0.06 - 6.20}{15}$$

which is taken to be zero since $\sigma_\phi^2 < 0$.

This procedure should now be repeated for the other locations and if for all these locations the hypothesis $H: \sigma_\phi^2 = 0$ is accepted, the procedure discussed in Section 8. 4 should now be applied to make the decision about the within flight variations and the between locations variations.

(b) Fixed Effects Model

If the hypothesis $H: \sigma_\phi^2 = 0$ had been rejected, the fixed effects model could have been employed to find out which of the four flights differed significantly. The result of the hypothesis $H: \phi_i = 0$ is the same for this example, but the change in concept now allows one only to draw conclusions about the four flights that were made. But now, that is all that is necessary.

One could now compute k from Eq. (8.8) and any two flight averages (e.g., the average vibration level of Flight 1 is 75.0/15 = 5 g's rms) differing by more than k are said to differ significantly.

Although this procedure of switching models may be somewhat useful in practice, one is on dangerous theoretical ground. Note that the switch to the fixed effects model concept is made if and only if $H: \phi_i = 0$ is rejected. But consider the situation if an error of the first kind has been made. This would mean that sampling error is being accepted as fact, and further tests are applied. It is clear that the same level of significance and power $(1 - \beta)$ for these subsequent tests cannot be what they seem on the surface. The sample being worked with has been "conditioned" by the first test, but the probabilities on which this second test is based do not take this into account.

8.3 TWO-WAY ANALYSIS OF VARIANCE, ONE OBSERVATION PER COMBINATION

As discussed previously, this procedure applies when measurements during several flights were made at one single point within the flight vehicle.

8.3.1 Mathematical Models

(a) Fixed Effects Model

Again, let x_{ij} be the jth vibration measurement of the ith flight. This measurement can be in terms of mean square acceleration, rms acceleration, peak amplitude, or any other type that may be required.

The mathematical model can then be written as

$$x_{ij} = \zeta_{ij} + \epsilon_{ij} \tag{8.12}$$

where ζ_{ij} is the jth mean vibration level during the ith flight (the expected value of x_{ij}) and ϵ_{ij} is a random effect which is assumed to be independently normally distributed with mean 0 and unknown variance σ^2. (See Tables 8.5 and 8.6.)

From Table 8.6 one can write for the mean ζ_{ij}

$$\zeta_{ij} = \zeta_{..} + \phi_i + \gamma_j + \eta_{ij} \tag{8.13}$$

where $\eta_{ij} = \zeta_{ij} - \zeta_{i.} - \zeta_{.j} + \zeta_{..}$ and is known as the interaction term.

		Vibration Levels				Totals
		1	2	j	c	
Flights	1	x_{11}	x_{12}	x_{1j}	x_{1c}	$\sum_{j=1}^{c} x_{1j} = R_1$
	2	x_{21}	x_{22}	x_{2j}	x_{2c}	$\sum_{j=1}^{c} x_{2j} = R_2$
	i	x_{i1}	x_{i2}	x_{ij}	x_{ic}	$\sum_{j=1}^{c} x_{ij} = R_i$
	r	x_{r1}	x_{r2}	x_{rj}	x_{rc}	$\sum_{j=1}^{c} x_{rj} = R_r$
		$C_1 = \sum_{i=1}^{r} x_{i1}$	$C_2 = \sum_{i=1}^{r} x_{i2}$	$C_j = \sum_{i=1}^{r} x_{ij}$	$C_c = \sum_{i=1}^{r} x_{ic}$	$T = \sum_{i=1}^{r} \sum_{j=1}^{c} x_{ij}$

Table 8. 5 Pairing of Flights and Vibration Levels

		Vibration Levels				Average over Levels	Differences
		1	2	j	c		
Flights	1	ζ_{11}	ζ_{12}	ζ_{1j}	ζ_{1c}	$\zeta_{1\cdot}$	$\phi_1 = \zeta_{1\cdot} - \zeta_{\cdot\cdot}$
	2	ζ_{21}	ζ_{22}	ζ_{2j}	ζ_{2c}	$\zeta_{2\cdot}$	$\phi_2 = \zeta_{2\cdot} - \zeta_{\cdot\cdot}$
	i	ζ_{i1}	ζ_{i2}	ζ_{ij}	ζ_{ic}	$\zeta_{i\cdot}$	$\phi_i = \zeta_{i\cdot} - \zeta_{\cdot\cdot}$
	r	ζ_{r1}	ζ_{r2}	ζ_{rj}	ζ_{rc}	$\zeta_{r\cdot}$	$\phi_r = \zeta_{r\cdot} - \zeta_{\cdot\cdot}$
Average over Flights		$\zeta_{\cdot 1}$	$\zeta_{\cdot 2}$	$\zeta_{\cdot j}$	$\zeta_{\cdot c}$	Over-all Average $\zeta_{\cdot\cdot}$	
Differ-ences		$\gamma_1 = \zeta_{\cdot 1} - \zeta_{\cdot\cdot}$	$\gamma_2 = \zeta_{\cdot 2} - \zeta_{\cdot\cdot}$		$\gamma_j = \zeta_{\cdot j} - \zeta_{\cdot\cdot}$	$\gamma_c = \zeta_{\cdot c} - \zeta_{\cdot\cdot}$	

Table 8. 6 Table of Expected Values of Vibration Levels

An interaction between two effects ϕ_i and γ_j is said to exist if the joint effect of the two taken together is different from the sum of the separate effects. For example, assume there are five machines and four operators, and it is desired to test whether the machines differ in the number of units produced per day. It is possible that the second operator works much better on the third machine than on the others. The resultant increase in production of the third machine, therefore, cannot be assumed to be a characteristic of the man only or the machine only, but is due to the "interaction" of that particular man with that particular machine.

From Eq. (8. 13) one sees that the mean for the ith flight and jth measurement is written as a constant $\zeta_{..}$ plus an effect due to the ith flight which is constant over all vibration levels, plus an effect due to the jth vibration level which is constant over all flights, plus an interaction term η_{ij}. Testing whether the flights are homogeneous is equivalent to testing

$$H : \phi_1 = \phi_2 = \ldots = \phi_r = 0 \tag{8. 14}$$

Testing whether the vibration levels from time period to time period are homogeneous is equivalent to testing

$$H : \gamma_1 = \gamma_2 = \ldots = \gamma_c = 0 \tag{8. 15}$$

and testing for no interaction is equivalent to testing

$$H : \eta_{ij} = 0 \tag{8. 16}$$

for all i and j.

It should again be emphasized that the inference drawn from the above tests <u>only</u> apply to the particular flights made and the vibration levels measured. Inferences about other similar future flights or vibration levels not measured cannot be made using the above model.

One other comment must be made concerning the interpretation of the analysis as a test for stationarity. Every flight must be, in concept, broken down in advance into a finite set of time periods. For example, if two-second records are being collected from a two-hour flight, think of the flight as 3600 time periods. Then certain of these time periods are selected (the same for all flights) and samples are drawn from these specific time periods for each flight. In the fixed effects model, no very satisfactory

interpretation of the test for equivalences of data from these different time periods is available. However, in the random effects models to be described, a subset of these time periods would be randomly selected. (These time periods would correspond to the random sampling plan which would of necessity be the same for every flight to allow this model to make sense.) Now, the test for equivalence of measurements from the various time periods can be interpreted in some cases as a test for stationarity. This is described in Section 16.

It should also be noted that since only one measurement is available for the ith row and jth column (as shown in Table 8.5), one cannot test for interaction and it has to be assumed that interaction does not exist.

(b) Random Effects Model

As indicated above, more useful results would be obtained if inferences could be drawn about all other similar flights and all possible measurements. Here it will be assumed that the four flights of the previous section were selected at random from many possible flights and that the measurements were made at random throughout each of the flights. The mathematical model can still be written as (combining Eqs. (8.12) and (8.13))

$$x_{ij} = \zeta_{..} + \phi_i + \gamma_j + \eta_{ij} + \epsilon_{ij} \qquad (8.17)$$

but the interpretation is different. The ϕ_i (between flight effects), the γ_j (within flight effects) and η_{ij} (interaction effects) are now independent random variables all having mean 0 and variance σ_ϕ^2, σ_γ^2, and σ_η^2 respectively. The random variable ϵ is assumed to be normally distributed. The random variables, ϕ, γ, and η are also often assumed to be normally distributed if further inferences about these components are to be made.

Here the inferences about row effects (between flights), column effects (within flights), and interaction is equivalent to making inferences about σ_ϕ^2, σ_γ^2, and σ_η^2 respectively. For example, if $\sigma_\phi^2 = 0$, all the flights are homogeneous.

(c) Mixed Effects Model

The previous two sections dealt with the two-way analysis of variance when both the Flights and Measurements were considered as fixed effects, or when both were considered as random effects. A third possibility is to consider the Flights as fixed effects and the Measurements as random effects, or, to consider the Flights as random effects and the Measurements as fixed effects. This model is denoted the "mixed effects model." It should be emphasized that the decision as to whether the flights or measurements are fixed or random effects is a practical one and is not up to the statistician. If the vibration engineer wishes to draw conclusions only about the measurements that were made, and does not want to generalize the results, the measurements are then considered fixed effects. This would be equivalent to making inferences about γ_j. If the engineer then wants to extend his conclusions to other flights not included in his experiment, the flights are then considered as random effects, but he has to make sure that the flights chosen were selected at random from all the possible flights. This would be equivalent to making inferences about σ_ϕ^2. The usefulness of this model will become clear in Section 8.3.3.

8.3.2 Computational Procedure

In order to keep track of all the calculations that have to be made, it is recommended that an "Analysis of Variance Table" be constructed similar to that shown in Table 8.7. To complete this table, the following computational procedure can be employed. The subscripts r and c will be used to denote the number of rows and columns, respectively. The relationships between the computational procedure and the individual measurements x_{ij} of Table 8.5 are shown in Table 8.7.

(1) Calculate row totals

$$R_1, R_2, \ldots, R_r$$

(2) Calculate column totals

$$C_1, C_2, \ldots, C_c$$

Source	Sum of Squares	Degrees of Freedom	Mean Square	Expected Mean Square for the Fixed Effects Model	Expected Mean Square for the Random Effects Model	Expected Mean Square for the Mixed Effects Model
Within Flights (Between Columns)	$SS_4 = r\sum_{j=1}^{c} (\bar{x}_{.j} - \bar{x}_{..})^2$	$c - 1$	$SS_4^* = \frac{SS_4}{(c-1)}$	$\sigma^2 + \frac{r}{c-1}\sum_{j=1}^{c} \gamma_j^2$	$\sigma^2 + \sigma_\eta^2 + r\sigma_\gamma^2$	$\sigma^2 + \sigma_\eta^2 + \frac{r}{c-1}\sum_{j=1}^{c} \gamma_j^2$
Between Flights (Between rows)	$SS_3 = c\sum_{i=1}^{r} (\bar{x}_{i.} - \bar{x}_{..})^2$	$r - 1$	$SS_3^* = \frac{SS_3}{(r-1)}$	$\sigma^2 + \frac{c}{r-1}\sum_{i=1}^{r} \phi_i^2$	$\sigma_2 + \sigma_\eta^2 + c\sigma_\phi^2$	$\sigma^2 + c\sigma_\phi^2$
Residual	$SS_2 = \sum_{i=1}^{r}\sum_{j=1}^{c} (x_{ij} - \bar{x}_{i.} - \bar{x}_{.j} + \bar{x}_{..})^2$	$(r-1)(c-1)$	$SS_2^* = \frac{SS_2}{(r-1)(c-1)}$	$\sigma^2 + \frac{1}{(r-1)(c-1)}\sum_{i=1}^{r}\sum_{j=1}^{c} \eta_{ij}^2$	$\sigma^2 + \sigma_\eta^2$	$\sigma^2 + \sigma_\eta^2$
Total	$SS = \sum_{i=1}^{r}\sum_{j=1}^{c} (x_{ij} - \bar{x}_{..})^2$	$rc - 1$	$SS^* = \frac{SS}{(rc-1)}$	—	—	—

$$\bar{x}_{.j} = \frac{1}{r}\sum_{i=1}^{r} x_{ij} = \frac{C_j}{r}; \quad \bar{x}_{i.} = \frac{1}{c}\sum_{j=1}^{c} x_{ij} = \frac{R_i}{c}; \quad \bar{x}_{..} = \frac{1}{rc}\sum_{i=1}^{r}\sum_{j=1}^{c} x_{ij} = \frac{T}{rc}$$

For example: $$SS_4 = r\sum_{j=1}^{c} (\bar{x}_{.j} - \bar{x}_{..})^2 = r\sum_{j=1}^{c} \left(\frac{C_j}{r} - \frac{T}{rc}\right)^2 = r\sum_{j=1}^{c} \left(\frac{C_j^2}{r^2} - \frac{2C_jT}{r^2c} + \frac{T^2}{r^2c^2}\right) = \sum_{j=1}^{c} \frac{C_j^2}{r} - \frac{2T^2}{rc} + \frac{T^2}{rc} = \sum_{j=1}^{c} \frac{C_j^2}{r} - \frac{T^2}{rc}$$

Table 8.7. Analysis of Variance Table, Two-Way Classification, One Observation per Combination

(3) Calculate over-all total

$$T = R_1 + R_2 + \ldots + R_r = C_1 + C_2 + \ldots + C_c$$

(This also provides a check for row and column totals.)

(4) Calculate crude total sum of squares

$$\sum_{i=1}^{r} \sum_{j=1}^{c} x_{ij}^2 = x_{11}^2 + x_{12}^2 + \ldots + x_{rc}^2$$

(5) Calculate crude sum of squares between rows

$$\sum_{i=1}^{r} \frac{R_i^2}{c} = \frac{\left(R_1^2 + R_2^2 + \ldots + R_r^2\right)}{c}$$

(6) Calculate crude sum of squares between columns

$$\sum_{j=1}^{c} \frac{C_j^2}{r} = \frac{\left(C_1^2 + C_2^2 + \ldots + C_c^2\right)}{r}$$

(7) Calculate Correction Factor due to mean

$$\text{C. F.} = \frac{T^2}{rc}$$

(8)
$$SS_4 = (6) - (7) = \sum_{j=1}^{c} \frac{C_j^2}{r} - \frac{T^2}{rc}$$

(9)
$$SS_3 = (5) - (7) = \sum_{i=1}^{r} \frac{R_i^2}{c} - \frac{T^2}{rc}$$

(10)
$$SS = (4) - (7) = \sum_{i=1}^{r} \sum_{j=1}^{c} x_{ij}^2 - \frac{T^2}{rc}$$

(11)
$$SS_2 = (10) - (9) - (8) = SS - SS_3 - SS_4$$

Now the Analysis of Variance Table can be completed by calculating the values of SS_4^*, SS_3^*, and SS_2^* as shown in Table 8.7.

(a) Decision Procedure, Fixed Effects Model

Before applying the decision procedure for the fixed effects model with only one observation per cell, it should be noted that the interaction term must be zero. As mentioned in Section 8. 3. 1, a test for interaction cannot be made; therefore, the designer of the experiment should make sure that the interaction can be assumed to be zero.

It was shown in Section 8. 3. 1 that testing the hypothesis about the homogeneity of flights is equivalent to testing the hypothesis

$$H: \phi_1 = \phi_2 = \ldots = \phi_r$$

It can be shown (Ref. [2], p. 323) that this hypothesis is true if the ratio SS_3^*/SS_2^* is distributed as an F random variable with(r - 1) and (r - 1)(c - 1) degrees of freedom. Therefore, the hypothesis of the equality of the ϕ's is accepted if

$$F = \frac{SS_3^*}{SS_2^*} \leq F_{\alpha;(r-1),(r-1)(c-1)} \tag{8.18}$$

Similarly, the hypothesis of equality of the γ's (no flight effects) is accepted if

$$F = \frac{SS_4^*}{SS_2^*} \leq F_{\alpha;(c-1),(r-1)(c-1)} \tag{8.19}$$

If the hypothesis that there are no between flight effects or no within flight effects or both is rejected, the Tukey k factor can again be used to determine which row or which column averages differ significantly. The equations are

$$k \text{ (for rows)} = k^* \sqrt{\frac{SS_2^*}{c}} \tag{8.20}$$

$$k \text{ (for columns)} = k^* \sqrt{\frac{SS_2^*}{r}} \tag{8.21}$$

Again, k^* is read from the tables in Appendix 8-A.The proper degrees of freedom are those corresponding to the degrees of freedom of the residual source in the analysis of variance table, i. e. , (r-1)(c-1).

(b) Decision Procedure, Random Effects Model

For this procedure, no assumption has to be made about the interaction being zero. In the random effects model, it was shown that testing the hypothesis about the row effects (between flights) is equivalent to testing the hypothesis $H: \sigma_{\phi}^{2} = 0$. It can be shown (Ref. [2], p. 327) that the ratio SS_3^*/SS_2^* is distributed as an F random variable with (r - 1) and (r - 1)(c - 1) degrees of freedom when the hypothesis is true. Therefore, the hypothesis that $\sigma_{\phi}^{2} = 0$ is accepted if

$$F = \frac{SS_3^*}{SS_2^*} \leq F_{\alpha;(r-1),(r-1)(c-1)} \tag{8.22}$$

Similarly, the hypothesis of no column effects (within flight effects), i.e., the hypothesis that $\sigma_{\gamma}^{2} = 0$ is accepted if

$$F = \frac{SS_4^*}{SS_2^*} \leq F_{\alpha;(c-1),(r-1)(c-1)} \tag{8.23}$$

In order to obtain a measure of how much the row effects or column effects differ, the quantities σ_{ϕ}^{2} and σ_{γ}^{2} can be estimated. These estimates are given by

$$\hat{\sigma}_{\phi}^{2} = \frac{SS_3^* - SS_2^*}{c} \tag{8.24}$$

$$\hat{\sigma}_{\gamma}^{2} = \frac{SS_4^* - SS_2^*}{r} \tag{8.25}$$

If any of these quantities is negative, the estimate is taken to be zero.

(c) Decision Procedure, Mixed Effects Model

In the mixed effects model, the row effects will be assumed as the random effects and the column effects will be assumed as the fixed effects. Therefore, testing the hypothesis about the homogeneity of row effects (between flight effects) is equivalent to testing the hypothesis $H: \sigma_{\phi}^{2} = 0$. Equation (8.22) can now be used to test the hypothesis and Eq. (8.24) can be used to estimate σ_{ϕ}^{2}

	Vibration Levels															Totals
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	
Flights 1	7	8	9	8	8	5	2	1.5	2.5	2	4	4.5	5	4.5	4	75.0
2	8	7	9	7	8	6	2.5	2	1.5	2	4	5	4	4.5	4.5	75.0
3	8	9	7	9	7	5	2.5	2.5	1.5	2	3.5	4	4.5	4	5	74.5
4	7	9	8	7	9	5	1.5	2	2	2.5	4	4	3.5	5	3.5	73.0
Total	30.0	33.0	33.0	31.0	32.0	21.0	8.5	8.0	7.5	8.5	15.5	17.5	17.0	18.0	17.0	297.5
Averages	7.5	8.25	8.25	7.75	8.0	5.25	2.12	2.0	1.88	2.12	3.88	4.38	4.25	4.50	4.25	

Table 8.8 Pairing of Flights and Vibration Levels

Testing for homogeneity of column effects (within flight effects) is equivalent to testing the hypothesis $H: \gamma_1 = \gamma_2 = \dots = \gamma_c = 0$. Equation (8. 19) can therefore be used to test for the hypothesis of equality of the γ's and if the hypothesis is rejected, Eq. (8. 21) can be used to determine which columns differ significantly.

There is one important difference here from the fixed effects model. Even though the column effects are fixed, no assumption has to be made about the interaction, i. e. , the presence of interaction does not invalidate the decision procedure for the mixed effects model.

8. 3. 3 Numerical Example

For simplicity, the same data will be used here as that used for the example in Section 8. 2. 2. This data is reproduced in Table 8. 8, showing the row totals, column totals, and the column averages.

$$\sum_{i=1}^{4} \sum_{j=1}^{15} x_{ij}^2 = 1,822.25$$

$$\sum_{i=1}^{4} \frac{R_i^2}{c} = \frac{22,129.25}{15} = 1,475.28$$

$$\sum_{j=1}^{15} \frac{C_j^2}{r} = \frac{7,217.25}{4} = 1,804.31$$

$$\frac{T^2}{rc} = \frac{88,506.25}{60} = 1,475.10$$

$$SS_4 = \sum_{j=1}^{15} \frac{C_j^2}{r} - \frac{T^2}{rc} = 329.21$$

$$SS_3 = \sum_{i=1}^{4} \frac{R_i^2}{c} - \frac{T^2}{rc} = 0.18$$

$$SS = \sum_{i=1}^{4} \sum_{j=1}^{15} x_{ij}^2 - \frac{T^2}{rc} = 347.15$$

$$SS_2 = SS - SS_3 - SS_4 = 17.76$$

The completed Analysis of Variance Table is shown in Table 8. 9.

Source	Sum of Squares	Degrees of Freedom	Mean Square
Within Flights	$SS_4 = 329.21$	$(c-1) = 14$	$SS_4^* = \frac{SS_4}{14} = 23.52$
Between Flights	$SS_3 = 0.18$	$(r-1) = 3$	$SS_3^* = \frac{SS_3}{3} = 0.06$
Residual	$SS_2 = 17.76$	$(r-1)(c-1) = 42$	$SS_2^* = \frac{SS_2}{42} = 0.42$
Total	$SS = 347.15$	$(rc-1) = 59$	$SS^* = \frac{SS}{59} = 5.88$

Table 8.9 Analysis of Variance Table

(a) Random Effects Model

Since both the flights and the measurements were made at random, and inferences are to be drawn about all the flights and the entire vibration environment of all these flights, the random effects model will be used. Choosing a level of significance $\alpha = 0.05$, the between flight homogeneity is tested by Eq. (8.22), namely the hypothesis $H: \sigma_\phi^2 = 0$ is accepted if

$$F = \frac{SS_3^*}{SS_2^*} = \frac{0.06}{0.42} \leq F_{0.05;3,42}$$

From the F tables (Ref. [2], p. 560) one obtains $F_{0.05;3,42} = 2.83$ and therefore accept the hypothesis, i.e., there are no significant differences between flights, since $F = 0.14$.

The homogeneity of the within flight vibration levels is tested by Eq. (8.23) Namely, the hypothesis $H: \sigma_Y^2 = 0$ is accepted if

$$F = \frac{SS_4^*}{SS_2^*} = \frac{23.52}{0.42} \leq F_{0.05;14,42}$$

From the F tables one obtains $F_{0.05;\,14,\,42} = 1.93$ and therefore rejects the hypothesis, i.e., there is a significant difference of vibration levels within flights, since $F = 56.0$.

Another interpretation of this result was mentioned in Section 8.3.1. That is, the vibration flight history of these flights from which these samples were taken is a nonstationary process. Therefore, the above analysis of variance technique provides one with a test for stationarity.

A measure of how much the within flight data varies is given by Eq. (8.25)

$$\hat{\sigma}_Y^2 = \frac{SS_4^* - SS_2^*}{r} = \frac{23.1}{4} = 5.78$$

Therefore an estimate of the standard deviation of the within flight data is

$$\hat{\sigma}_Y = \sqrt{\sigma_Y^2} = 2.40 \text{ g's rms}$$

It should be noted that by looking at the original data the within flight variations appear to follow the flight phases. Very high vibration levels were recorded during the take-off phase, low levels were recorded during the cruise phase, and intermediate levels were recorded during the landing phase. The above procedure tested whether the differences of vibration levels between phases were significant. Since they were significant, no conclusions can now be drawn about the within flight vibration levels as a whole.

However it appears that if a separate analysis of variance is performed for each flight phase, the hypothesis of homogeneity might be accepted. The problem now arises to determine where one flight phase leaves off and the next one begins. A quantitative method is available to do this. Since the division is to be made only for the vibration levels that were recorded, a mixed effects model can be assumed.

(b) Mixed Effects Model

The hypothesis now has become $H: \gamma_1 = \gamma_2 = \cdots = \gamma_{15} = 0$. As previously discussed, the F statistic is the same. However, the change in concept allows us now to determine (with the cautions previously mentioned) which particular vibration levels of the 15 differ significantly.

The Tukey k factor given by Eq. (8.21) can now be used. For this example $k^* = 5.11$ (from the tables in Appendix 8-A), and

$$k = 5.11\sqrt{\frac{0.42}{4}} = 1.66$$

From Table 8.8 one can see that measurements 1 through 5 can be considered as one phase, measurements 7 through 10 as one phase, and measurements 11 through 15 as one phase. Measurement number 6 should be lumped with numbers 1 through 5 since it is closer to the take-off phase than the cruise phase. The analysis of variance can now be repeated for the four flights, using only one flight phase at a time. To draw conclusions about each particular phase for all flights, the random effects model should be used.

A note of caution is due at this point. From the data in Table 8.8 it would appear that the flight phases could have been separated just by inspection. However, many times, data that appears to differ actually does not differ significantly in the statistical sense, or data that appears to be uniform may be rejected as homogeneous by the proper statistical techniques.

8.4 TWO-WAY ANALYSIS OF VARIANCE, SEVERAL OBSERVATIONS PER COMBINATION

In the previous section, vibration measurements were made only at one point within the flight vehicle for each of the flights. However, it may be desired to make measurements for each flight at several points simultaneously. For example, electronic equipment is to be installed in the pilot's compartment, in the mid-section, and near the tail of the aircraft. Vibration measurements would then be made at each of these locations. The question now arises, can one establish an over-all vibration specification for all three locations or will a separate specification be required for each of the three locations. This will require a decision whether or not the vibration levels at the three locations can be considered as statistically equivalent.

First, a one-way analysis of variance has to be made for each of the three locations separately, as discussed in Section 8.2, to test for between flight differences. If the between flight measurements for each of the locations turns out to be not statistically equivalent, obviously no further analysis is required since conclusions about other flights cannot be drawn. If, however, flights can be considered as equivalent, one can now determine if the locations and the within flight differences are statistically equivalent.

8.4.1 Mathematical Models

A table should be constructed as that shown in Table 8.10. In Table 8.10, x_{ijv} is the ith measurement, at the jth location for the vth flight. It should be noted in Table 8.10 that the symbols r, c, and n always refer to the total number of rows, columns, and observations per combination, regardless of what they represent.

The reader should also note that if, as discussed above, the three locations were deliberately chosen, they will always have to be considered as fixed effects, and all inferences drawn apply only to those three locations. If the flights and the measurements were chosen at random, it should be clear that a mixed effects model should be used. For convenience in the mixed effects model, the columns are usually designated as the fixed effects. Of course, if the locations were chosen at random,

Locations →

Measurements ↓	1	2	------	c
1	x_{111} x_{112} ⋮ x_{11n}	x_{121} x_{122} x_{12n}	x_{1j1} x_{1j2} x_{1jn}	x_{1c1} x_{1c2} ⋮ x_{1cn}
2	x_{211} x_{212} ⋮ x_{21n}	-- -- --		
⋮	x_{i11} x_{i12} ⋮ x_{i1n}		x_{ijv}	
r	x_{r11} x_{r12} ⋮ x_{r1n}			x_{rc1} x_{rc2} ⋮ x_{rcn}

Table 8. 10 Arrangement of Data for Two-Way Analysis of Variance, n Observations per Combination

the random effects model would apply and conclusions could then be drawn about locations where measurements were not made. It should also be noted that since more than one observation is available per combination, the experimenter can now test for interaction.

The general mathematical model now is as follows:

$$x_{ijv} = \zeta_{..} + \phi_i + \gamma_j + \eta_{ij} + \epsilon_{ijv} \quad ; \quad \begin{aligned} i &= 1, 2, \ldots, r \\ j &= 1, 2, \ldots, c \\ v &= 1, 2, \ldots, n \end{aligned} \tag{8.26}$$

where $\zeta_{..}$ is the general mean and ϵ_{ijv} are the experimental errors which are assumed to be independently normally distributed each with mean 0 and variance σ^2.

(a) Fixed Effects Model

In the fixed effects model, ϕ_i is the effect of adding the ith fixed measurements $\sum_{i=1}^{r} \phi_i = 0$, γ_j is the effect of adding the jth fixed location $\sum_{j=1}^{c} \gamma_i = 0$, and η_{ij} is the fixed interaction of the ith measurement with the jth location.

(b) Random Effects Model

In the random effects model, ϕ_i is the effect of adding the ith measurements and is a random variable and is usually assumed to be normally distributed with mean 0 and variance σ_ϕ^2. γ_j is the effect of adding the jth location and is a random variable usually assumed to be normally distributed with mean 0 and variance σ_γ^2.

η_{ij} is the interaction of the ith measurements with the jth location and is also a random variable usually assumed to be normally distributed with mean 0 and variance σ_η^2.

(c) Mixed Effects Model

Here ϕ_i is the effect of adding the ith measurement and is a random variable assumed to be normally distributed with mean 0 and variance σ_ϕ^2.

γ_i is the effect of adding the jth fixed location, $\sum_{j=1}^{c} \gamma_i = 0$.

η_{ij} is the interaction of the ith random measurement with the jth fixed location and is a random variable usually assumed to be normally distributed with mean 0 and variance σ_η^2.

8.4.2 Computational Procedure

An analysis of variance table as shown in Table 8.11 will now have to be completed. The following computational procedures can be used for this purpose:

(1) Calculate row totals: $R_1, R_2, \ldots, R_r$

(2) Calculate column totals: $C_1, C_2, \ldots, C_c$

Source	Sum of Squares	Degrees of Free-dom	Mean Square	Expected Mean Square for the Fixed Effects Model	Expected Mean Square for the Random Effects Model	Expected Mean Square for the Mixed Effects Model
Between Columns (Between Locations)	$SS_4 = rn\sum_{j=1}^{c}(\bar{x}_{.j.} - \bar{x}_{...})^2$	c-1	$SS_4^* = \frac{SS_4}{(c-1)}$	$\sigma^2 + \frac{rn}{c-1}\sum_{j=1}^{c}\gamma_j^2$	$\sigma^2 + n\sigma_\eta^2 + rn\sigma_\gamma^2$	$\sigma^2 + n\sigma_\eta^2 + \frac{rn}{c-1}\sum_{j=1}^{c}\gamma_j^2$
Between Rows (Within Flights)	$SS_3 = cn\sum_{i=1}^{r}(\bar{x}_{i..} - \bar{x}_{...})^2$	r-1	$SS_3^* = \frac{SS_3}{(r-1)}$	$\sigma^2 + \frac{cn}{r-1}\sum_{i=1}^{r}\phi_i^2$	$\sigma^2 + n\sigma_\eta^2 + cn\sigma_\phi^2$	$\sigma^2 + nc\sigma_\phi^2$
Interaction	$SS_2 = n\sum_{i=1}^{r}\sum_{j=1}^{c}(\bar{x}_{ij.} - \bar{x}_{i..} - \bar{x}_{.j.} + \bar{x}_{...})^2$	(r-1)(c-1)	$SS_2^* = \frac{SS_2}{(r-1)(c-1)}$	$\sigma^2 + \frac{n}{(r-1)(c-1)}\sum_{i=1}^{r}\sum_{j=1}^{c}\eta_{ij}^2$	$\sigma^2 + n\sigma_\eta^2$	$\sigma^2 + n\sigma_\eta^2$
Within Combinations (Between Flights)	$SS_1 = \sum_{i=1}^{r}\sum_{j=1}^{c}\sum_{v=1}^{n}(x_{ijv} - \bar{x}_{ij.})^2$	rc(n-1)	$SS_1^* = \frac{SS_1}{rc(n-1)}$	σ^2	σ^2	σ^2
Total	$SS = \sum_{i=1}^{r}\sum_{j=1}^{c}\sum_{v=1}^{n}(x_{ijv} - \bar{x}_{...})^2$	rcn-1	$SS^* = \frac{SS}{(rcn-1)}$	—	—	—

In this table $\bar{x}_{.j.} = \sum_{i=1}^{r}\sum_{v=1}^{n} x_{ijv}/rn$; $\bar{x}_{i..} = \sum_{j=1}^{c}\sum_{v=1}^{n} x_{ijv}/n$; $\bar{x}_{...} = \sum_{i=1}^{r}\sum_{j=1}^{c}\sum_{v=1}^{n} x_{ijv}/rcn$

Table 8.11. Analysis of Variance Table, Two-Way Classification, n Observations per Combination

(3) Calculate within combination totals: $W_{11}, W_{12}, \ldots, W_{rc}$

(4) Calculate over-all total:

$$T = R_1 + R_2 + \ldots + R_r = C_1 + C_2 + \ldots + C_c$$

(5) Calculate crude sum of squares:

$$\sum_{i=1}^{r} \sum_{j=1}^{c} \sum_{v=1}^{n} x_{ijv}^2 = x_{111}^2 + x_{112}^2 + \ldots + x_{rcn}^2$$

(6) Calculate crude sum of squares between columns

$$\sum_{i=1}^{c} \frac{C_j^2}{rn} = \frac{\left(C_1^2 + C_2^2 + \ldots + C_c^2\right)}{rn}$$

(7) Calculate crude sum of squares between rows

$$\sum_{i=1}^{r} \frac{R_i^2}{cn} = \frac{\left(R_1^2 + R_2^2 + \ldots + R_r^2\right)}{cn}$$

(8) Calculate crude sum of squares between combinations

$$\sum_{i=1}^{r} \sum_{j=1}^{c} \frac{W_{ij}^2}{n} = \frac{\left(W_{11}^2 + W_{12}^2 + \ldots + W_{rc}^2\right)}{n}$$

(9) Calculate correction factor due to mean

$$C.\,F. = \frac{T^2}{rcn}$$

From the quantities above compute:

(10) $SS_4 = (6) - (9) = \sum_{j=1}^{c} \frac{C_j^2}{rn} - \frac{T^2}{rcn}$

(11) $SS_3 = (7) - (9) = \sum_{i=1}^{r} \frac{R_i^2}{cn} - \frac{T^2}{rcn}$

(12) $SS_1 = (5) - (8) = \sum_{i=1}^{r} \sum_{j=1}^{c} \sum_{v=1}^{n} x_{ijv}^2 - \sum_{i=1}^{r} \sum_{j=1}^{c} \frac{W_{ij}^2}{n}$

(13) $SS = (5) - (9) = \sum_{i=1}^{r} \sum_{j=1}^{c} \sum_{v=1}^{n} x_{ijv}^{2} - \frac{T^2}{rcn}$

(14) $SS_2 = SS - SS_1 - SS_3 - SS_4$

The mean square values SS^*, SS_1^*, SS_2^*, SS_3^*, and SS_4^* can now be calculated by dividing by the appropriate number of degrees of freedom as shown in Table 8. 11.

(a) Decision Procedure, Fixed Effects Model

The first hypothesis that should be tested, when the fixed effects model is used, is for no interaction. This is equivalent to testing $H: \eta_{ij} = 0$ for all i and j. It can be shown (Ref. [2], p. 336) that if the hypothesis of no interaction is true, SS_2^*/SS_1^* is distributed as an F random variable with (r-1)(c-1) and rc(n-1) degrees of freedom. Therefore, the hypothesis $\eta_{ij} = 0$ is accepted if

$$F = \frac{SS_2^*}{SS_1^*} \leq F_{a;(r-1)(c-1), rc(n-1)} \qquad (8.27)$$

If this hypothesis is rejected, the tests for significance of the within flight effects (row effects), and between locations effects (column effects), are now meaningless. If the hypothesis that there is no interaction is accepted, tests for row effects and column effects can be made.

It can be shown (Ref. [2], p. 338) that if the hypothesis about no row effects is true, SS_3^*/SS_1^* is distributed as an F random variable with (r-1) and rc(n-1) degrees of freedom. Therefore, the hypothesis $H: \phi_1 = \phi_2 = \ldots = \phi_r = 0$ is accepted if

$$F = \frac{SS_4^*}{SS_1^*} \leq F_{a;(r-1), rc(n-1)} \qquad (8.28)$$

Similarly the hypothesis of no column effects, i. e., $H: \gamma_1 = \gamma_2 = \ldots = \gamma_c = 0$ is accepted if

$$F = \frac{SS_4^*}{SS_1^*} \leq F_{a;(c-1), rc(n-1)} \qquad (8.29)$$

If the row or column effects are significant, the Tukey k factor can again be used to determine which rows or columns differ significantly.

$$k \text{ (for rows)} = k^* \sqrt{\frac{SS_1^*}{nc}} \qquad (8.30)$$

$$k \text{ (for columns)} = k^* \sqrt{\frac{SS_1^*}{nr}} \qquad (8.31)$$

The factor k^* is read from the table in Appendix 8-A. The proper degrees of freedom are those corresponding to the degrees of freedom of the within combination source in the analysis of variance table, i. e., $rc(n-1)$.

(b) <u>Decision Procedure, Random Effects Model</u>

In this model, a test for interaction, i. e., $H: \sigma_\eta^2 = 0$, is also available. The hypothesis of no interaction, i. e., the hypothesis that $\sigma_\eta^2 = 0$ is accepted if

$$F = \frac{SS_2^*}{SS_1^*} \leq F_{\alpha;(r-1)(c-1),\, rc(n-1)} \qquad (8.32)$$

In this model, even if interaction is present, one can still test for row or column effects. The hypothesis of no row effects (within flight effects), i. e., the hypothesis that $\sigma_\phi^2 = 0$ is accepted if

$$F = \frac{SS_3^*}{SS_2^*} \leq F_{\alpha;(r-1),\,(r-1)(c-1)} \qquad (8.33)$$

The hypothesis of no column effects (between location effects), i. e., the hypothesis that $\sigma_\gamma^2 = 0$ is accepted if

$$F = \frac{SS_4^*}{SS_2^*} \leq F_{\alpha;(c-1),\,(r-1)(c-1)} \qquad (8.34)$$

As before, the quantities σ_η^2, σ_ϕ^2, and σ_γ^2 can be estimated, namely

$$\hat{\sigma}_\eta^2 = \frac{SS_2^* - SS_1^*}{n} \tag{8.35}$$

$$\hat{\sigma}_\phi^2 = \frac{SS_3^* - SS_2^*}{cn} \tag{8.36}$$

$$\hat{\sigma}_\gamma^2 = \frac{SS_4^* - SS_2^*}{rn} \tag{8.37}$$

If any of these quantities is negative the estimate is taken to be zero.

(c) Decision Procedure, Mixed Effects Model

In the mixed effects model the row effects will be assumed as the random effects and the column effects will be assumed as the fixed effects.

In this model a test for interaction is also available. The hypothesis that there is no interaction, i. e., $H: \sigma_\eta^2 = 0$ is accepted if

$$F = \frac{SS_2^*}{SS_1^*} \leq F_{\alpha;(r-1)(c-1),\, rc(n-1)} \tag{8.38}$$

Again, even if interaction is present, one can still test for row or column effects. The hypothesis of no row effects, i. e., $H: \sigma_\phi^2 = 0$ is accepted if

$$F = \frac{SS_3^*}{SS_1^*} \leq F_{\alpha;(r-1),\, rc(n-1)} \tag{8.39}$$

An estimate of σ_ϕ^2 is obtained from

$$\hat{\sigma}_\phi^2 = \frac{SS_3^* - SS_1^*}{nc} \tag{8.40}$$

The hypothesis of no column effects, i.e., $H: \gamma_1 = \gamma_2 = \dots = \gamma_c = 0$ is accepted if

$$F = \frac{SS_4^*}{SS_2^*} \leq F_{\alpha;(c-1),(r-1)(c-1)} \tag{8.41}$$

The Tukey k factor is given by

$$k = k^* \sqrt{\frac{SS_2^*}{nr}} \tag{8.42}$$

and can again be used to determine which columns differ significantly. The proper degrees of freedom are those corresponding to the degrees of freedom of the interaction source in the analysis of variance table, i.e., $(r-1)(c-1)$.

8.4.3 Numerical Example

It will be assumed that all measurements were made at random during four flights which were selected at random, at three specific locations within each flight vehicle. Location 1 is near the tail of the aircraft, location 2 near the middle of the aircraft, and location 3 in the pilot's compartment. During each flight and at each location, 15 samples were taken at random. Three main flight phases could be identified by visual inspection of the data and were taken to be 1) take-off and climb, 2) cruise, and 3) descent and landing. It is desired to predict the over-all vibration environment at those three locations since electronic equipment is to be installed. Before making this prediction, the following questions have to be answered:

1. Were the four flights statistically equivalent so that the data can be pooled?
2. Can one over-all prediction be made or should a separate prediction be made for each of the three locations?
3. Can the prediction be made for the entire flight or will the flight have to be divided into flight phases, and if so, how?

If the answer is "no" to the first question, one can go no further. If the answer is "yes", questions 2 and 3 can now be answered.

The data is in terms of g's rms, each number representing the over-all rms value of one sample, two seconds long. The bandwidth of the analyzing instrument was 2000 cps so that the assumption of

normality for the distribution of the rms values is a good approximation, since the number of degrees of freedom for each sample is quite large.

The data is as follows:

Location 1:

Flight 1: 7, 8, 9, 8, 8, 5, 2, 1.5, 2.5, 2, 4, 4.5, 5, 4.5, 4
Flight 2: 8, 7, 9, 7, 8, 6, 2.5, 2, 1.5, 2, 4, 5, 4, 4.5, 4.5
Flight 3: 8, 9, 7, 9, 7, 5, 2.5, 2.5, 1.5, 2, 3.5, 4, 4.5, 4, 5
Flight 4: 7, 9, 8, 7, 9, 5, 1.5, 2, 2, 2.5, 4, 4, 3.5, 5, 3.5

Location 2:

Flight 1: 5, 6, 7, 6, 6, 3, 1, 0.5, 1.5, 1, 2, 2.5, 3, 2.5, 2
Flight 2: 6, 5, 7, 5, 6, 4, 1.5, 1. 0.5, 1, 2, 3, 2, 2.5, 2.5
Flight 3: 6, 7, 5, 7, 5, 3, 1.5, 1.5, 0.5, 1.0, 2.5, 2, 2.5, 2, 3
Flight 4: 5, 7, 6, 5, 7, 3, 0.5, 1, 1, 1.5, 3, 2, 2.5, 3, 2.5

Location 3:

Flight 1: 4, 5, 6, 5, 5, 2, 1, 0.5, 1.5, 1, 2, 1.5, 2, 1.5, 1
Flight 2: 5, 4, 6, 4, 5, 3, 1.5, 1, 0.5, 1, 2, 2, 1, 1.5, 1.5
Flight 3: 5, 6, 4, 6, 4, 2, 1.5, 1.5, 0.5, 1.0, 1.5, 1, 1.5, 1, 2
Flight 4: 4, 6, 5, 4, 6, 2, 0.5, 1, 1, 1.5, 2, 1, 1.5, 2, 1.5

For convenience, the data for location 1 is identical to the data used in the previous example. In Section 8.2.3 it was shown that the between flight differences were not significant. A similar one-way analysis of variance has to be performed for locations 2 and 3. This analysis will not be shown here, but the reader can verify for himself that the between flight difference for locations 2 and 3 are also not significant.

The next step is to determine if between location and within flight differences are significant. Since the measurements were made at random but the locations were picked deliberately, a mixed effects model will have to be used.

(a) Mixed Effects Model

The data has been rewritten in tabular form and is shown in Table 8.12. Table 8.13 was prepared from Table 8.12, first computing within combination totals, then computing row and column totals. Using the computational procedure of Section 8.4.2 results in the following:

(1) Row totals are shown in Table 8. 13

(2) Column totals are shown in Table 8. 13

(3) Within combination totals are shown in Table 8. 13

(4) Over-all total is shown in Table 8. 13

(5) Crude sum of squares is obtained from Table 8. 12

$$\sum_{i=1}^{r} \sum_{j=1}^{c} \sum_{v=1}^{n} x_{ijv}^{2} = 3,322.75$$

(6) Crude sum of squares between columns

$$\sum_{j=1}^{c} \frac{C_j^2}{rn} = \frac{151,692.75}{60} = 2,528.21$$

(7) Crude sum of squares between rows

$$\sum_{i=1}^{r} \frac{R_i^2}{cn} = \frac{36,861.25}{12} = 3,071.77$$

(8) Crude sum of squares between combinations

$$\sum_{i=1}^{r} \sum_{j=1}^{c} \frac{W_{ij}^2}{n} = \frac{13,085.75}{4} = 3,271.44$$

(9) $$\text{C. F.} = \frac{T^2}{rcn} = \frac{423,150.25}{180} = 2,350.83$$

(10) $SS_4 = (6) - (9) = 2,528.21 - 2,350.83 = 177.38$

(11) $SS_3 = (7) - (9) = 3,071.77 - 2,350.83 = 720.94$

(12) $SS_1 = (5) - (8) = 3,322.75 - 3,271.44 = 51.31$

(13) $SS = (5) - (9) = 3,322.75 - 2,350.83 = 971.92$

(14) $SS_2 = SS - SS_1 - SS_3 - SS_4 = 22.29$

Measurements	Locations 1		Locations 2		Locations 3	
1	7	8	5	6	4	5
	8	7	6	5	5	4
2	8	7	6	5	5	4
	9	9	7	7	6	6
3	9	9	7	7	6	6
	7	8	5	6	4	5
4	8	7	6	5	5	4
	9	7	7	5	6	4
5	8	8	6	6	5	5
	7	9	5	7	4	6
6	5	6	3	4	2	3
	5	5	3	3	2	2
7	2	2.5	1	1.5	1	1.5
	2.5	1.5	1.5	0.5	1.5	0.5
8	1.5	2	0.5	1	0.5	1
	2.5	2	1.5	1	1.5	1
9	2.5	1.5	1.5	0.5	1.5	0.5
	1.5	2	0.5	1	0.5	1
10	2	2	1	1	1	1
	2	2.5	1.0	1.5	1.0	1.5
11	4	4	2	2	2	2
	3.5	4	2.5	3	1.5	2
12	4.5	5	2.5	3	1.5	2
	4	4	2	2	1	1
13	5	4	3	2	2	1
	4.5	3.5	2.5	2.5	1.5	1.5
14	4.5	4.5	2.5	2.5	1.5	1.5
	4	5	2	3	1	2
15	4	4.5	2	2.5	1	1.5
	5	3.5	3	2.5	2	1.5

Table 8.12 Pairing of Locations and Measurements

		Locations			Totals	Averages
		1	2	3		
Measurements	1	30.0	22.0	18.0	70.0	5.83
	2	33.0	25.0	21.0	79.0	6.58
	3	33.0	25.0	21.0	79.0	6.58
	4	31.0	23.0	19.0	73.0	6.08
	5	32.0	24.0	20.0	76.0	6.33
	6	21.0	13.0	9.0	43.0	3.58
	7	8.5	4.5	4.5	17.5	1.46
	8	8.0	4.0	4.0	16.0	1.33
	9	7.5	3.5	3.5	14.5	1.21
	10	8.5	4.5	4.5	17.5	1.46
	11	15.5	9.5	7.5	32.5	2.71
	12	17.5	9.5	5.5	32.5	2.71
	13	17.0	10.0	6.0	33.0	2.75
	14	18.0	10.0	6.0	34.0	2.83
	15	17.0	10.0	6.0	33.0	2.75
	Totals	297.5	197.5	155.5	$\boxed{650.5}$	---
	Averages	4.96	3.29	2.59	---	---

Table 8.13 Pairing of Locations and Measurements, Pooling the Data of Four Flights for Each Combination

From these values an analysis of variance table as shown in Table 8.11 can be constructed. The completed table is shown in Table 8. 14.

Source	Sum of Squares	Degrees of Freedom	Mean Squares
Between columns (between locations)	$SS_4 = 177.38$	$(c-1) = 2$	$SS_4^* = \frac{SS_4}{c-1} = 88.69$
Between rows (within flights)	$SS_3 = 720.94$	$(r-1) = 14$	$SS_3^* = \frac{SS_3}{r-1} = 51.49$
Interaction	$SS_2 = 22.29$	$(r-1)(c-1) = 28$	$SS_2^* = \frac{SS_2}{(r-1)(c-1)} = 0.80$
Within Combinations (between flights)	$SS_1 = 51.31$	$rc(n-1) = 135$	$SS_1^* = \frac{SS_1}{rc(n-1)} = 0.38$
Total	$SS = 971.92$	$rcn-1 = 179$	$SS^* = \frac{SS}{rcn-1} = 5.43$

Table 8. 14 Analysis of Variance Table

Choosing a level of significance $\alpha = 0.05$, the hypothesis of no interaction, i. e., $H: \sigma_{\eta}^2 = 0$ is accepted if (from Eq. 8. 38)

$$F = \frac{SS_2^*}{SS_1^*} = \frac{0.80}{0.38} = 2.10 \leq F_{.05;28,135}$$

From the F tables, $F_{.05;28,135} = 1.57$ and therefore the hypothesis is rejected. In other words, there is interaction between measurements and locations within the aircraft.

The hypothesis about between location effects, i. e., $H: \gamma_1 = \gamma_2 = \gamma_3 = 0$ is accepted if (from Eq. 8. 41)

$$F = \frac{SS_4^*}{SS_2^*} = \frac{88.69}{0.80} = 110.86 \leq F_{.05;2,28}$$

From the F tables, $F_{.05;2,28} = 3.34$ and, therefore, reject the hypothesis. In other words, there are significant differences between locations within the aircraft. To determine which locations differ, the Tukey k factor is calculated from Eq. (8.42)

$$k = k^{*}\sqrt{\frac{SS_2^{*}}{nr}} = 3.50\sqrt{\frac{0.80}{60}} = 0.4$$

Comparing the column averages shown in Table 8.13 results in the conclusion that all three locations differ significantly since all the column averages differ by more than 0.4. Therefore, a separate prediction should be made for each location.

The hypothesis about the between measurement effects, i.e., $H: \sigma_{\phi}^{2} = 0$ is accepted if (from Eq. 8.39)

$$F = \frac{SS_3^{*}}{SS_1^{*}} = \frac{51.49}{.38} = 135.50 \leq F_{.05;14,135}$$

From the F tables, $F_{.05;14,135} = 1.77$ and, therefore, reject the hypothesis. In other words, there are significant differences in the within flight measurements.

The variance σ_{ϕ}^{2} can be estimated from Eq. (8.40) and is

$$\hat{\sigma}_{\phi}^{2} = \frac{SS_3^{*} - SS_1^{*}}{nc} = \frac{51.49 - 0.38}{12} = 4.26$$

An estimate of the standard deviation of the within flight measurements would be

$$\sigma_{\phi} = \sqrt{4.27} = 2.07 \text{ g's rms}$$

Since the measurements for each entire flight are not equivalent, each flight should be broken up into flight phases. Since this discussion will be made using only the particular measurements that were taken, the rows will now also be interpreted as fixed effects. This change in concept will allow the use of the Tukey k factor to determine which rows differ significantly.

The mathematical model has now become a fixed effects model and, therefore, Eq. (8. 30) should be used. From Eq. (8. 30)

$$k = k^{*}\sqrt{\frac{SS_1^{*}}{nc}} = k^{*}\sqrt{\frac{0.38}{12}} = 0.18\,k^{*}$$

From Appendix A, since 15 effects are being studied and SS_1^* has 135 degrees of freedom, $k^* \approx 4.90$. Therefore

$$k = (0.18)(4.90) = .88$$

Any two row averages differing by more than 0. 88 are said to differ significantly. From Table 8. 13, rows 1 through 5, 7 through 10, and 11 through 15 are therefore equivalent. A judgment will now have to be made as to what to do with row 6. This row will be considered as part of flight phase 2 (cruise).

The analysis of variance should now be repeated for each of the flight phases. For purposes of illustration, phase number 1 will be chosen. The data for phase 1 is shown in Table 8. 15.

The computational procedure yields the following:

(1) Row totals shown in Table 8. 15

(2) Column totals shown in Table 8. 15

(3) Within combination totals shown in Table 8. 15

(4) Over-all total shown in Table 8. 15

(5) Crude sum of squares

$$\sum_{i=1}^{r} \sum_{j=1}^{c} \sum_{v=1}^{n} x_{ijv}^{2} = 2,501.00$$

(6) Crude sum of squares between columns

$$\sum_{j=1}^{c} \frac{C_j^2}{rn} = \frac{49,243}{20} = 2,462.15$$

(7) Crude sum of squares between rows

$$\sum_{i=1}^{r} \frac{R_i^2}{cn} = \frac{28,487}{12} = 2,373.92$$

	Locations			Totals	Averages
	1	2	3		
Measurements 1	7 8 8 7 $\boxed{30}$	5 6 6 5 $\boxed{22}$	4 5 5 4 $\boxed{18}$	70	5.83
2	8 7 9 9 $\boxed{33}$	6 5 7 7 $\boxed{25}$	5 4 6 6 $\boxed{21}$	79	6.58
3	9 9 7 8 $\boxed{33}$	7 7 5 6 $\boxed{25}$	6 6 4 5 $\boxed{21}$	79	6.58
4	8 7 9 7 $\boxed{31}$	6 5 7 5 $\boxed{23}$	5 4 6 4 $\boxed{19}$	73	6.08
5	8 8 7 9 $\boxed{32}$	6 6 5 7 $\boxed{24}$	5 5 4 6 $\boxed{20}$	76	6.33
Totals	159	119	99	$\boxed{377}$	
Averages	7.95	5.95	4.95		

Table 8.15 Pairing of Measurements and Locations for Phase 1

(8) Crude sum of squares between combinations

$$\sum_{i=1}^{r} \sum_{j=1}^{c} \frac{W_{ij}^2}{n} = \frac{9,869}{4} = 2,467.25$$

(9)

$$C.F. = \frac{T^2}{rcn} = \frac{142,129}{60} = 2368.82$$

(10) $SS_4 = (6) - (9) = 93.33$

(11) $SS_3 = (7) - (9) = 5.10$

(12) $SS_1 = (5) - (8) = 33.75$

(13) $SS = (5) - (9) = 132.18$

(14) $SS_2 = SS - SS_1 - SS_3 - SS_4 = 0$

The completed analysis of variance table is shown in Table 8. 16.

Source	Sum of Squares	Degrees of Freedom	Mean Squares
Between columns (between locations)	$SS_4 = 93.33$	$(c-1) = 2$	$SS_4^* = 46.66$
Between rows (within flights)	$SS_3 = 5.10$	$(r-1) = 4$	$SS_3^* = 1.28$
Interaction	$SS_2 = 0$	$(r-1)(c-1)=8$	$SS_2^* = 0$
Within combinations (between flights)	$SS_1 = 33.75$	$rc(n-1) = 45$	$SS_1^* = 0.75$
Total	$SS = 132.18$	$rcn-1 = 59$	$SS^* = 2.24$

Table 8. 16 Analysis of Variance Table

Again using the decision procedure for the mixed effects model, the hypothesis that there is no interaction, i. e., $H: \sigma_\eta^2 = 0$ is accepted if

$$F = \frac{SS_2^*}{SS_1^*} \leq F_{\alpha;8,45}$$

from Eq. (8. 38). Since $SS_2^* = 0$, the hypothesis is accepted, i. e., there is no interaction.

The hypothesis of no row effects, i. e., $H: \sigma_\phi^2 = 0$ is accepted if

$$F = \frac{SS_3^*}{SS_1^*} = \frac{1.28}{0.75} = 1.71 \leq F_{\alpha;4,45}$$

Choosing a level of significance $\alpha = .05$, the F tables show $F_{.05;4,45} = 2.58$. Therefore, the hypothesis that there is no significant difference between rows is accepted.

An estimate of σ_ϕ^2 is given by Eq. (8. 40), namely,

$$\hat{\sigma}_\phi^2 = \frac{SS_3^* - SS_1^*}{nc} = \frac{1.28 - 0.75}{12} = 0.044$$

An estimate of σ_ϕ is

$$\hat{\sigma}_\phi = \sqrt{0.044} = 0.21 \text{ g's rms}$$

The hypothesis of no column effects, i.e., $H: \gamma_1 = \gamma_2 = \gamma_3 = 0$ is accepted from Eq. (8.41) if

$$F = \frac{SS_4^*}{SS_2^*} = \frac{46.66}{0} = \infty < F_{\alpha;2,8}$$

(Note that the term SS_2^* is actually very small but only zero due to computational round-off. The number 46.66/0 is, of course, undefined.) Obviously this hypothesis is rejected. The Tukey k factor from Eq. (8.42) is

$$k = k^* \sqrt{\frac{SS_2^*}{nr}} = k^* \sqrt{\frac{0}{nr}} = 0$$

Therefore, all three columns' averages differ significantly.

The over-all result is that 1) the flights are equivalent, 2) the measurements for phase 1 are equivalent, and 3) the locations are not equivalent. Therefore, a separate over-all prediction for each location can be made, pooling the data from all flights for phase 1. The reader should verify for himself that similar conclusions can be reached for phase 2 and phase 3.

8.5 REFERENCES

1. Bendat, J. S., Enochson, L. D., Klein, G. H., and A. G. Piersol, The Application of Statistics to the Flight Vehicle Vibration Problem, ASD TR 61-123, Aeronautical Systems Division, Air Force Systems Command, USAF, Wright-Patterson AFB, Ohio. 1961. ASTIA AD 271 913.

2. Bowker, H. A., and G. J. Lieberman, Engineering Statistics, Prentice-Hall, Inc., Englewood Cliffs, New Jersey. 1961.

Degrees of Freedom	2	3	4	5	6	7	8	9	10
	No. of Row or Column Effects Being Studied								
1	17·97	26·98	32·82	37·08	40·41	43·12	45·40	47·36	49·07
2	6·08	8·33	9·80	10·88	11·74	12·44	13·03	13·54	13·99
3	4·50	5·91	6·82	7·50	8·04	8·48	8·85	9·18	9·46
4	3·93	5·04	5·76	6·29	6·71	7·05	7·35	7·60	7·83
5	3·64	4·60	5·22	5·67	6·03	6·33	6·58	6·80	6·99
6	3·46	4·34	4·90	5·30	5·63	5·90	6·12	6·32	6·49
7	3·34	4·16	4·68	5·06	5·36	5·61	5·82	6·00	6·16
8	3·26	4·04	4·53	4·89	5·17	5·40	5·60	5·77	5·92
9	3·20	3·95	4·41	4·76	5·02	5·24	5·43	5·59	5·74
10	3·15	3·88	4·33	4·65	4·91	5·12	5·30	5·46	5·60
11	3·11	3·82	4·26	4·57	4·82	5·03	5·20	5·35	5·49
12	3·08	3·77	4·20	4·51	4·75	4·95	5·12	5·27	5·39
13	3·06	3·73	4·15	4·45	4·69	4·88	5·05	5·19	5·32
14	3·03	3·70	4·11	4·41	4·64	4·83	4·99	5·13	5·25
15	3·01	3·67	4·08	4·37	4·59	4·78	4·94	5·08	5·20
16	3·00	3·65	4·05	4·33	4·56	4·74	4·90	5·03	5·15
17	2·98	3·63	4·02	4·30	4·52	4·70	4·86	4·99	5·11
18	2·97	3·61	4·00	4·28	4·49	4·67	4·82	4·96	5·07
19	2·96	3·59	3·98	4·25	4·47	4·65	4·79	4·92	5·04
20	2·95	3·58	3·96	4·23	4·45	4·62	4·77	4·90	5·01
24	2·92	3·53	3·90	4·17	4·37	4·54	4·68	4·81	4·92
30	2·89	3·49	3·85	4·10	4·30	4·46	4·60	4·72	4·82
40	2·86	3·44	3·79	4·04	4·23	4·39	4·52	4·63	4·73
60	2·83	3·40	3·74	3·98	4·16	4·31	4·44	4·55	4·65
120	2·80	3·36	3·68	3·92	4·10	4·24	4·36	4·47	4·56
∞	2·77	3·31	3·63	3·86	4·03	4·17	4·29	4·39	4·47

Degrees of Freedom	11	12	13	14	15	16	17	18	19	20
		No. of Row or Column Effects Being Studied								
1	50·59	51·96	53·20	54·33	55·36	56·32	57·22	58·04	58·83	59·56
2	14·39	14·75	15·08	15·38	15·65	15·91	16·14	16·37	16·57	16·77
3	9·72	9·95	10·15	10·35	10·52	10·69	10·84	10·98	11·11	11·24
4	8·03	8·21	8·37	8·52	8·66	8·79	8·91	9·03	9·13	9·23
5	7·17	7·32	7·47	7·60	7·72	7·83	7·93	8·03	8·12	8·21
6	6·65	6·79	6·92	7·03	7·14	7·24	7·34	7·43	7·51	7·59
7	6·30	6·43	6·55	6·66	6·76	6·85	6·94	7·02	7·10	7·17
8	6·05	6·18	6·29	6·39	6·48	6·57	6·65	6·73	6·80	6·87
9	5·87	5·98	6·09	6·19	6·28	6·36	6·44	6·51	6·58	6·64
10	5·72	5·83	5·93	6·03	6·11	6·19	6·27	6·34	6·40	6·47
11	5·61	5·71	5·81	5·90	5·98	6·06	6·13	6·20	6·27	6·33
12	5·51	5·61	5·71	5·80	5·88	5·95	6·02	6·09	6·15	6·21
13	5·43	5·53	5·63	5·71	5·79	5·86	5·93	5·99	6·05	6·11
14	5·36	5·46	5·55	5·64	5·71	5·79	5·85	5·91	5·97	6·03
15	5·31	5·40	5·49	5·57	5·65	5·72	5·78	5·85	5·90	5·96
16	5·26	5·35	5·44	5·52	5·59	5·66	5·73	5·79	5·84	5·90
17	5·21	5·31	5·39	5·47	5·54	5·61	5·67	5·73	5·79	5·84
18	5·17	5·27	5·35	5·43	5·50	5·57	5·63	5·69	5·74	5·79
19	5·14	5·23	5·31	5·39	5·46	5·53	5·59	5·65	5·70	5·75
20	5·11	5·20	5·28	5·36	5·43	5·49	5·55	5·61	5·66	5·71
24	5·01	5·10	5·18	5·25	5·32	5·38	5·44	5·49	5·55	5·59
30	4·92	5·00	5·08	5·15	5·21	5·27	5·33	5·38	5·43	5·47
40	4·82	4·90	4·98	5·04	5·11	5·16	5·22	5·27	5·31	5·36
60	4·73	4·81	4·88	4·94	5·00	5·06	5·11	5·15	5·20	5·24
120	4·64	4·71	4·78	4·84	4·90	4·95	5·00	5·04	5·09	5·13
∞	4·55	4·62	4·68	4·74	4·80	4·85	4·89	4·93	4·97	5·01

Table 8A-1 Table of Factors k* (5% Significance Level)[1] ([1], p. 296)

[1] This table is adapted with permission from Biometrika (1959, p. 465), "Upper 5% Points of the Studentized Range" (London: Biometrika Office)

9. THE RESPONSE OF NONLINEAR SYSTEMS TO RANDOM EXCITATION

9.1 INTRODUCTION

The subject of response of nonlinear systems to random excitation has received some attention in the literature only within the past twenty odd years. Because many physical systems exhibit non-linear behavior and the assumptions of linearity often produce large errors, this subject is becoming more and more important. Most of the recent results are scattered in many different technical journals and cover several fields of application. This report assembles some of this material in one place and presents the reader with an outline of what has been accomplished to date, where he can find additional information, and those areas that still need considerable attention.

9.2 HARDENING SPRING – SINGLE DEGREE-OF-FREEDOM SYSTEM

9.2.1 General Theory

Before discussing the response of nonlinear systems to random excitation, a brief review will be given of the response of linear systems to sinusoidal and random excitation since much of the discussion that follows will refer to these cases.

The general equation of motion for the system shown in Figure 9.1 is:

$$m\ddot{x} + c\dot{x} + F(x) = f(t) \tag{9.1}$$

where m is the mass, $\ddot{x}$ the acceleration, $\dot{x}$ the velocity, c the viscous damping coefficient, F(x) the restoring force in the spring, and f(t) the excitation force.

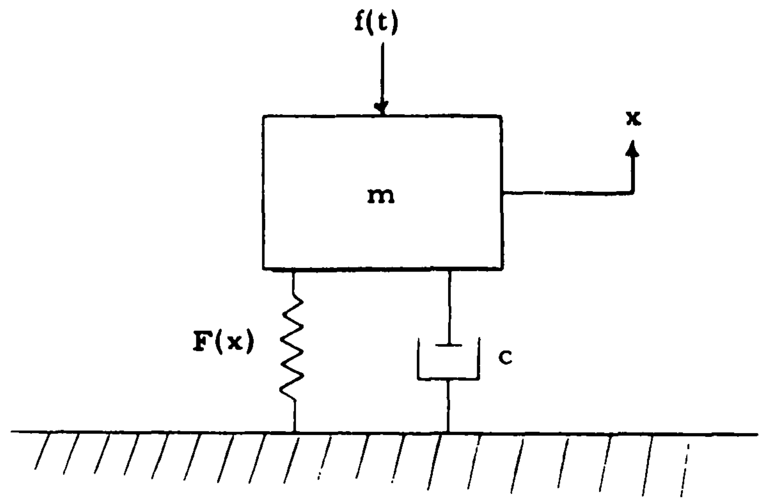


Figure 9.1 Single Degree-of-Freedom System With Force Applied to the Mass m.

If the motion is applied to the base, as shown in Figure 9.2, the equation of motion becomes

$$m\ddot{z} + c\dot{z} + F(z) = -m\ddot{u} \qquad (9.2)$$

where $z = x - u$ is the relative motion between the base and the mass m.

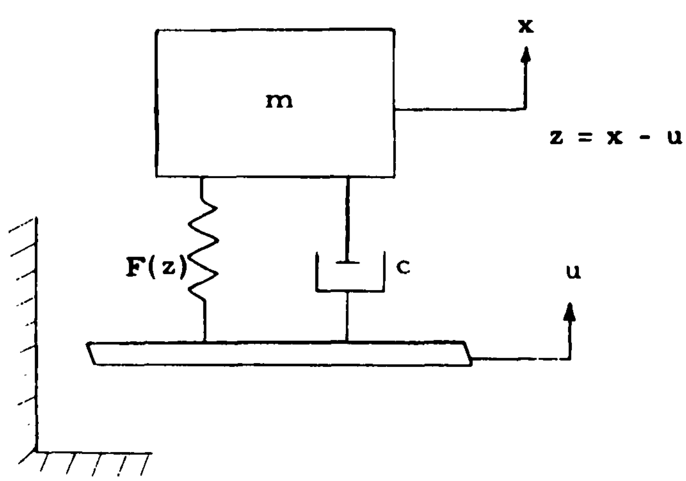


Figure 9.2. Single Degree-of-Freedom System With Motion Applied to the Base.

where $|H(\omega)|$ is the magnitude response function, or magnification factor, as defined by Ref. [15, p. 2-9].

$$|H(\omega)| = \sqrt{\frac{1}{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta \frac{\omega}{\omega_n}\right)^2}} \tag{9.6}$$

If the input is given as an acceleration $\ddot{u} = \ddot{u}_0 \sin \omega t$, then the acceleration of the mass m is

$$\ddot{x} = T\ddot{u} \tag{9.7}$$

and the relative acceleration is

$$\ddot{z} = |H(\omega)|\left(\frac{\omega^2}{\omega_n}\right)\ddot{u} \tag{9.8}$$

If it is desired to obtain the relative displacement z when the input is given in terms of the acceleration $\ddot{u}$ then the relationship becomes

$$z = |A(\omega)|\ddot{u} \tag{9.9}$$

where $|A(\omega)|$ is given by Ref. [15, p. 11-9]

$$|A(\omega)| = \frac{1}{\omega_n^2}\sqrt{\frac{1}{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta \frac{\omega}{\omega_n}\right)^2}} = \frac{|H(\omega)|}{\omega_n^2} \tag{9.10}$$

It should be noted that for the system shown in Figure 9.1 the absolute amplitude x of the mass m is equal to the relative amplitude z since the base is stationary (i.e., u = 0). If a force $f(t) = \mathbf{F}_i = \mathbf{F}_0 \sin \omega t$ is

The system described by Eq.(9.1) will be a linear system if and only if $F(z) = kz$ where k is a constant, and m and c are also constants. In the next section 9.2.2, $F(z)$ will be assumed to have this linear form. In the following section 9.2.3, $F(z)$ will be assumed to satisfy a nonlinear relation $F(z) = kz + rz^3$.

9.2.2 Linear System

Sinusoidal Input

If the restoring force in the spring of the system in Figure 9.2 is given by $F(z) = kz$ and the motion of the base is a displacement $u = u_0 \sin \omega t$, then the absolute displacement x of the mass m (ignoring phase) is given by

$$x = T_r u \tag{9.3}$$

where T_r is the transmissibility as defined by Ref. [15, p. 2-13].

$$T_r = \sqrt{\frac{1 + \left(\frac{2\zeta\omega}{\omega_n}\right)^2}{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}} \tag{9.4}$$

Here

$$\zeta = \frac{c}{2\sqrt{km}} \quad \text{and} \quad \omega_n = \sqrt{\frac{k}{m}}$$

The relative displacement z is given by

$$z = |H(\omega)| \left(\frac{\omega}{\omega_n}\right)^2 u \tag{9.5}$$

applied to the mass m in Figure 9.1, the force transmitted to the foundation F_T is given by

$$F_{T_r} = T_r F_i \tag{9.11}$$

where T_r is as defined in Eq.(9.4). The displacement x will then be

$$x = F_i \frac{1}{k} \left| H(\omega) \right| \tag{9.12}$$

and the acceleration $\ddot{x}$ will be

$$\ddot{x} = F_i \frac{\omega^2}{k} \left| H(\omega) \right| \tag{9.13}$$

where $|H(\omega)|$ is as defined by Eq.(9.6).

Random Inputs

When the input u in Figure 9.2 is random, the output displacements x and z can be calculated easily if ζ is small ($\zeta < 0.1$) and the input power spectral density $G_i(\omega)$, defined for positive frequencies only, is relatively constant in the neighborhood of the natural frequency of the system. See Figure 9.3, Ref.[11, p. 85].

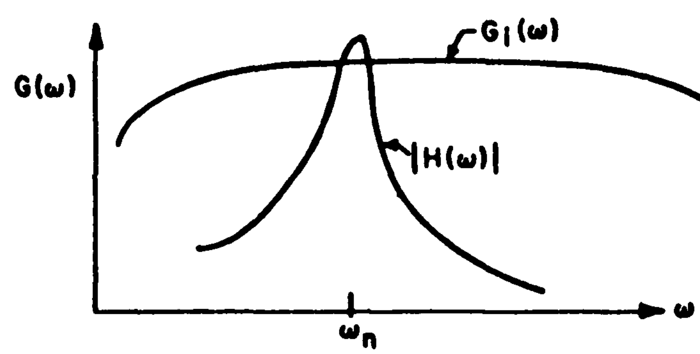


Figure 9.3. Typical Input Power Spectrum and Frequency Response Function vs. Frequency.

If one lets $G_o(\omega)$ be the output power spectrum, then the general relationship is Ref. [8, p. 82].

$$G_o(\omega) = |H(\omega)|^2 G_i(\omega) \approx |H(\omega)|^2 G_i(\omega_n) \qquad (9.14)$$

where $G_i(\omega)$ has been replaced by $G_i(\omega_n)$, a constant. This is justified when $|H(\omega)|$ has a sharp resonance at ω_n as shown in Figure 9.3. If the input is an acceleration power spectral density, then the output is also in terms of acceleration density and represents the power spectral density of the relative acceleration.

If the displacement output power spectrum is desired, when the input is an acceleration power spectrum, then the transfer function $|A(\omega)|$ as defined in Eq.(9.10) has to be used. In place of Eq(9.14), one obtains

$$G_{od}(\omega) = |A(\omega)|^2 G_{ia}(\omega_n) \qquad (9.15)$$

where $G_{od}(\omega)$ is the output displacement power spectral density and $G_{ia}(\omega)$ the input acceleration power spectral density.

The mean square output displacement (relative) is now given by

$$\begin{aligned} \overline{z^2} &= \int_0^\infty G_{od}(\omega)\, d\omega = \int_0^\infty G_{ia}(\omega_n)\, |A(\omega)|^2\, d\omega \\ &= \frac{\pi G_{ia}(\omega_n)}{4\zeta\, \omega_n^{\,3}} \end{aligned} \qquad (9.16)$$

If the input power spectrum G_{ia} is given in terms of cps rather than radians per seconds, let $G_{ia}(f) = 2\pi\, G_{ia}(\omega)$, and Eq(9.16) becomes

$$\overline{z^2} = \frac{G_{ia}(f_n)}{8\zeta\, \omega_n^{\,3}} \qquad (9.17)$$

The mean square output acceleration is obtained by the use of $|H(\omega)|$ in Eq.(9.16), which results in

$$\overline{\ddot{z}^2} = \frac{\pi\omega_n G_{ia}(\omega_n)}{4\zeta} = \frac{\omega_n G_{ia}(f_n)}{8\zeta} \tag{9.18}$$

If the input is given in units of force squared per radian $\left[G_{iF}(\omega)\right]$ applied to the mass m in Figure 9.1, the output displacement power spectral density is

$$G_{od}(\omega) = \frac{1}{k^2}\left|H(\omega)\right|^2 G_{iF}(\omega_n) \tag{9.19}$$

The mean square output displacement $\overline{z^2}$ is then

$$\overline{z^2} = \frac{\pi\omega_n G_{iF}(\omega_n)}{4\zeta\, k^2} = \frac{\omega_n G_{iF}(f_n)}{8\zeta\, k^2} \tag{9.20}$$

Another parameter of interest is the probability density of instantaneous amplitudes. If the input is Gaussian, the output will also be Gaussian and the density function is given by

$$p(z) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-z^2/2\sigma^2} \tag{9.21}$$

where $\sigma^2 = \overline{z^2}$ and the mean response $\bar{z}$ is assumed to be zero.

The response of a lightly damped ($\zeta < 0.1$) single degree-of-freedom system when subjected to random excitation is shown in Figure 9.4. The response curve represents a quasi-sine wave with frequency at approximately ω_n and a randomly varying amplitude. The average number of zero crossings per unit time would then be

$$\overline{\nu_0} = \frac{\omega_n}{\pi} \tag{9.22}$$

and the average number of zero crossings with positive slope $\bar{\nu}_0^+$ is

$$\bar{\nu}_0^+ = \frac{\omega_n}{2\pi} \tag{9.23}$$

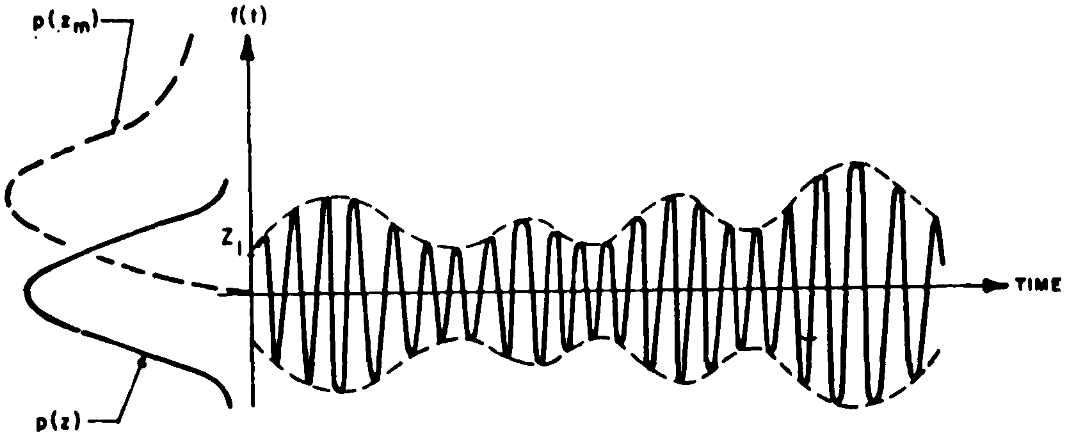


Figure 9.4. Response of Lightly Damped System to Random Excitation.

The distribution of the envelope is given by the Rayleigh probability density function Ref. [2].

$$p(z_m) = \frac{z_m}{\sigma^2} e^{-z_m^2/2\sigma^2} \tag{9.24}$$

The probability that z will exceed a specified value z_1 is determined by Ref. [2].

$$p\left[|z| > z_1\right] = \text{erfc}\, \frac{z_1}{\sigma\sqrt{2}} \approx \sqrt{\frac{2}{\pi}}\left(\frac{\sigma}{z_1}\right) e^{-z_1^2/2\sigma^2}\left[1 - \left(\frac{\sigma}{z_1}\right)^2 + \ldots\right] \qquad (9.25)$$

where erfc denotes the complementary error function and the approximation holds for $z_1 \gg \sigma$. Equation (9.25) can be used to determine the probability of exceeding some ultimate stress (in terms of z_1), whereas the Rayleigh distribution can be used for fatigue studies since then the probability distribution of peaks is of importance.

9.2.3 Nonlinear System, Wide-Band Excitation

The previous discussion has shown that the analysis of linear systems can usually be made without simplifying restrictions and exact solutions can be obtained for many parameters as long as the physical system is governed by the assumed equations of motion.

For nonlinear systems the situation is much less satisfactory. Even when a nonlinear system can be described exactly by its differential equation of motion, exact solutions cannot usually be obtained. One is therefore saddled with two levels of approximation. First, the differential equation represents an approximation to the actual system under study, and second, one has to use approximate methods to solve the equations.

Prior to obtaining any analytical solutions one can deduce in a qualitative manner what characteristics the response amplitude should have. Considering the single-degree-of-freedom system as a narrow-band oscillator, the response of a linear system is essentially that of a modulated sine wave oscillating at the natural frequency of the system. For a

hardening spring one would therefore expect an increase in this frequency and a decrease in the probability of large amplitudes by flattening of the peaks due to the continuous increase in stiffness of the system for increasing deflections. The discussion that follows will bear out these intuitive results.

To date, mostly two methods have been applied to the single degree-of-freedom system with a nonlinear restoring force. They will be discussed below, followed by a comparison of the two methods.

(a) Exact Solution

The first method uses the Fokker-Planck equation to obtain the joint probability density for the response velocity and displacement Ref. [20].

From Eq. (9.2),

$$m\ddot{z} + c\,\dot{z} + F\,(z) = -m\ddot{u}$$

Dividing by m, letting $\alpha = \frac{c}{2m}$, and $-\ddot{u} = A\,(t)$

$$\ddot{z} + 2\alpha\,\dot{z} + \frac{1}{m}\,F\,(z) = A(t) \tag{9.26}$$

If the random input A(t) is considered as an infinitely dense superposition of independent infinitesimal impulses, the joint probability density of displacement and velocity of the response, $p\,(z,\,\dot{z})$, can be obtained by solving the Fokker-Planck equation, Ref. [20].

$$\frac{\partial p}{\partial t} + \dot{z}\,\frac{\partial p}{\partial z} - \frac{\partial}{\partial \dot{z}}\left[2\alpha\dot{z} + \frac{1}{m}\;F\,(z)\right]p = \frac{1}{2}\;S_{ia}(f_n)\,\frac{\partial p}{\partial \dot{z}} \tag{9.27}$$

where $S_{ia}(f_n)$ is the acceleration power spectral density of A(t) over both positive and negative frequencies in terms of cps. Assuming the

input has been present since $t = -\infty$, one can let $\partial p/\partial t = 0$ and obtain, Ref. [20].

$$p(z, \dot{z}) = C \exp \left[\frac{-2\alpha}{S_{ia}(f)} \left(\dot{z}^2 + \frac{2}{m} \int F(z)\, dz \right) \right] \tag{9.28}$$

where C is a normalizing constant which can be determined by requiring

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(z, \dot{z})\, dz\, d\dot{z} = 1 \tag{9.29}$$

Nonlinear Cubic Elasticity Case (Hardening Spring)

Mindlin, Ref. [28], has given equations for several types of hardening springs. The main type analyzed to date has been the cubic elasticity equation which is

$$F(z) = k z + r z^3 \tag{9.30}$$

where k is the initial spring constant and r determines the rate of increase of z. Both Mindlin, Ref. [28], and Klein, Ref. [17], have given examples for the evaluation of k and r from experimental data. Equation (9.30) is plotted in Figure 9.5. Since F(z) is an odd function of z, even powers of z are not present.

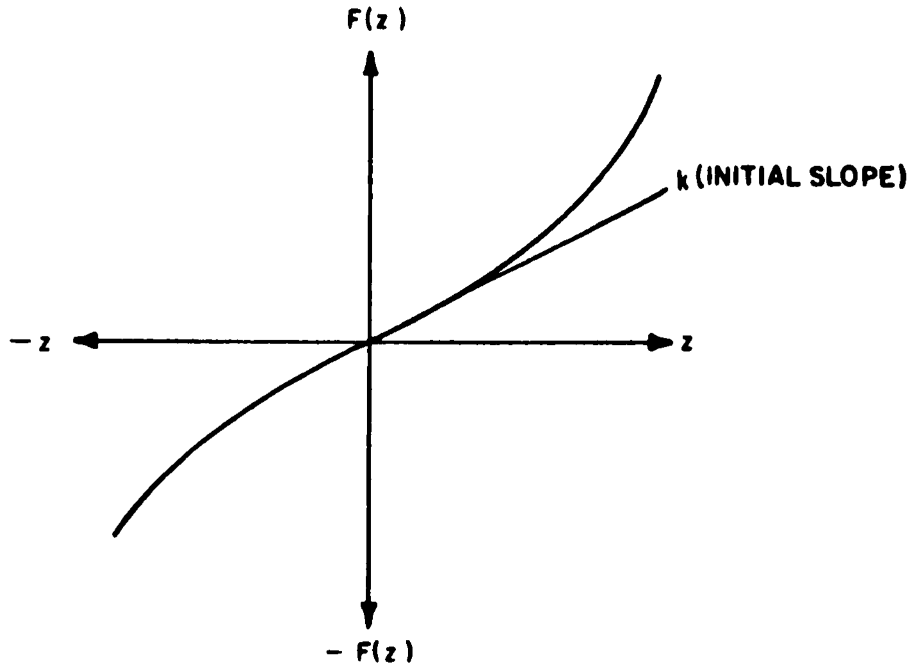


Figure 9.5. Cubic Elasticity Force - Deflection Curve
$$F(z) = kz + rz^3$$

The material to follow in Section 9.2.3 applies only to this special type of nonlinearity exhibited by Eq. (9.30).

Letting $r = \frac{k}{\ell^2}$* and $\omega_n^2 = \frac{k}{m}$, one can write

$$\frac{1}{m} F(z) = \omega_n^2 \left(z + \frac{z^3}{\ell^2} \right) \qquad (9.31)$$

Substituting Eq(9.31) into Eq.(9.28) and integrating the exponent one gets

$$p(z, \dot{z}) = C \exp \left\{ - \frac{2\alpha}{S_{ia}(f_n)} \left[\dot{z}^2 + \omega_n^2 \left(z^2 + \frac{z^4}{2\ell^2} \right) \right] \right\} \qquad (9.32)$$

*The parameter ℓ^2 was chosen for convenience since for the analysis of a pinned beam, ℓ is equal to twice the radius of gyration of the beam's section.

Since $S_{ia}(f)$ was defined such that both negative and positive frequencies contribute to the total mean square (i.e., $G_{ia}(f) = 2S_{ia}(f)$), one obtains from Eq. (9.17)

$$\sigma_r^2 = \frac{S_{ia}(f_n)}{4\zeta\omega_n^3} \tag{9.33}$$

where σ_r^2 is the mean square displacement if the system were linear, (i.e., $\sigma_r^2 = \overline{z^2}$). It is often convenient to use a dimensionless response. Let

$$\sigma_d^2 = \frac{\sigma_r^2}{\ell^2} \quad \text{and} \quad \zeta = \frac{a}{\omega_n}$$

Eq(9.33) becomes

$$\sigma_d^2 = \frac{S_{ia}(f_n)}{4a\omega_n^2\ell^2} \tag{9.34}$$

Substituting for $S_{ia}(f_n)$ in Eq.(9.32), and since $p(z, \dot{z})$ can be factored in terms of z and $\dot{z}$ making z and $\dot{z}$ statistically independent, one can write

$$p(z, \dot{z}) = p(z)\, p(\dot{z}) = C \exp\left[\frac{-\dot{z}^2}{2\sigma_d^2\,\omega_n^2\,\ell^2}\right] \exp\left[\frac{-z^2}{2\sigma_d^2\ell^2}\left(1 + \frac{z^2}{2\ell^2}\right)\right] \tag{9.35}$$

It is convenient to express the displacement z in terms of a normalized displacement $y = \frac{z}{\ell}$. Equation (9.35) then becomes

$$p(y)\, p(\dot{y}) = C \exp\left[\frac{-\dot{y}^2}{2\sigma_d^2\,\omega_n^2}\right] \exp\left[\frac{-y^2}{2\sigma_d^2}\left(1 + \frac{y^2}{2}\right)\right] \tag{9.36}$$

After solving for C (involving numerical integration), it has been shown that the velocity $\dot{y}$ is then normally distributed with mean value zero and variance $(\sigma_d \omega_n)^2$, Ref. [20], namely,

$$p(\dot{y}) = \frac{1}{\sqrt{2\pi}\ \sigma_d \omega_n} \exp\left[\frac{-\dot{y}^2}{2\sigma_d^{\ 2} \omega_n^{\ 2}}\right] \tag{9.37}$$

The displacement y, however, has the distribution, Ref. [24],

$$p(y) = \frac{\sqrt{2\pi}\ \overline{\nu}_0^+}{\omega_n \sigma_d} \exp\left[\frac{-y^2\left(1 + \frac{y^2}{2}\right)}{2\sigma_d^{\ 2}}\right] \tag{9.38}$$

where $\overline{\nu}_0^+$ is the number of zero crossings with positive slope and defined by the equation, Ref. [9].

$$\overline{\nu}_0^+ = \int_0^{\infty} \dot{z}\ p(0, \dot{z})\ d\dot{z} = C\ \frac{S_{ia}(f_n)}{4a} \tag{9.39}$$

if it is assumed that maxima occur only for values of $z > 0$ and minima occur only for values of $z < 0$. The quantity ν_0^+ can also be interpreted as an "average resonant frequency" for the nonlinear system if a is small. The ratio $f_n / \overline{\nu}_0^+$, where $f_n = \omega_n / 2\pi$ is the linear natural frequency obtained by letting $f(z) = kz$, has been evaluated by Lyon, Ref. [24], for various values of the linear mean-square response and is shown by the solid line in Figure 9.6, obtained from Ref. [20].

Figure 9.6 indicates the expected increase in $\overline{\nu}_0^+$ with an increase of the input. The increase of the resonant frequency of such a nonlinear system when subjected to sinusoidal excitation is indicated by the dashed line in Figure 9.6, obtained from Ref. [20].

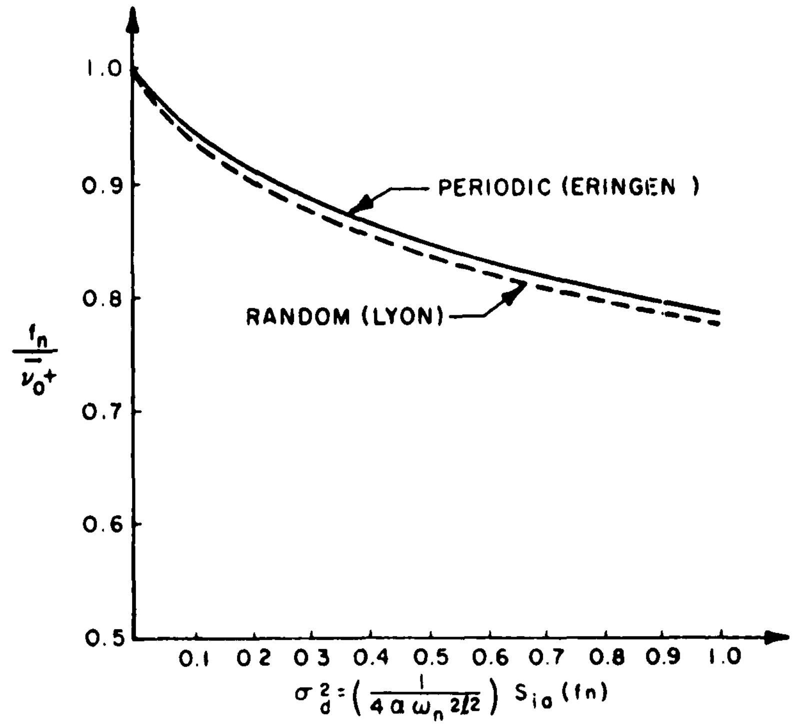


Figure 9. 6. Ratio of Linear Natural Frequency to Nonlinear "Average Resonant Frequency" vs. the Normalized Linear Mean-Square Displacement Response, σ_d^2.

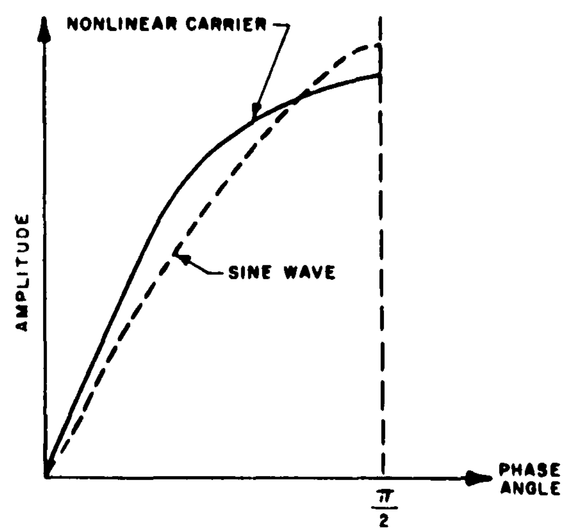


Figure 9. 7. Waveform of Nonlinear Carrier Compared to a Sine Wave

Lyon, Ref. [20], has also calculated the shape of the carrier wave and the general result is shown in Figure 9.7 which indicates the expected flattening of the amplitude peaks. It is beyond the scope of this report to discuss the analytical solution and the reader is referred to the references. For a discussion of probability density of peaks, see Section 9.3.4.

A curve of the ratio of nonlinear to linear mean-square response versus the linear mean-square response (which is a function of the input power spectrum) is shown in Figure 9.8, Ref. [21].

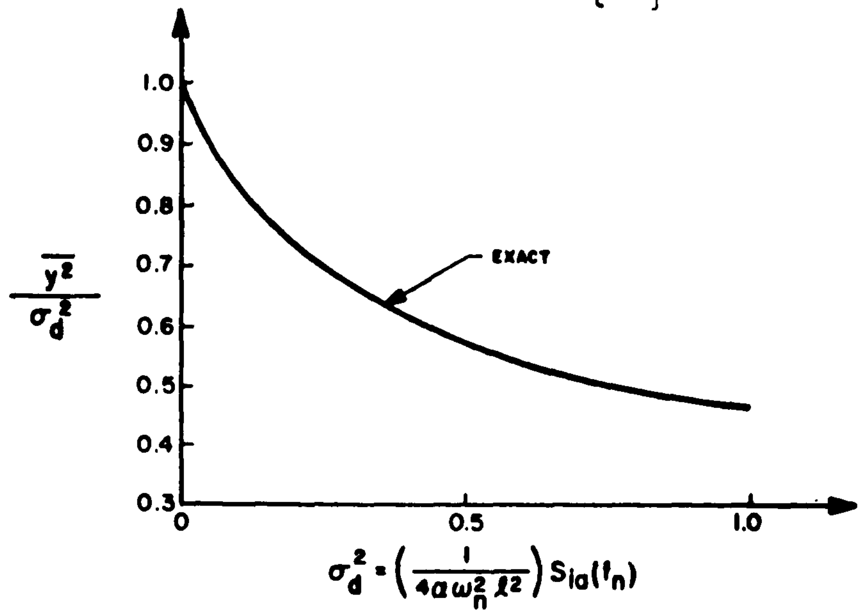


Figure 9.8. Ratio of Nonlinear to Linear Mean-Square Response vs. Linear Mean-Square Response

This result is obtained from the exact solution of $\overline{y^2}$ using the probability density of p(y) of Eq. (9.38), namely

$$\overline{y^2} = \int_{-\infty}^{\infty} y^2 \, p(y) \, dy \tag{9.40}$$

As can be seen from Figure 9.8, a considerable reduction occurs in the nonlinear mean-square displacement response compared to the linear response even for small values of the input.

(b) Approximate Solutions

In order to avoid the tedious job of numerical integration, approximations can be made in the original differential equation which result in much simpler methods of solution. One of the most widely used is the method of "Equivalent Linearization." See Ref. [4].

Substituting Eq.(9.31) into Eq.(9.26), one obtains

$$\ddot{z} + 2\alpha\dot{z} + \omega_n^2\left(z + \frac{z^3}{\ell^2}\right) = A(t) \tag{9.41}$$

Normalizing z by letting $y = \frac{z}{\ell}$, one can write

$$\ddot{y} + 2\alpha\dot{y} + \omega_n^2(y + y^3) = A'(t) \tag{9.42}$$

Adding the term λy to both sides, Eq. (9.42) can be written

$$\ddot{y} + 2\alpha\dot{y} + \lambda y = A'(t) - \omega_n^2(y + y^3) + \lambda y \tag{9.43}$$

where

$$-\omega^2(y + y^3) + \lambda y = R_e \tag{9.44}$$

is called the remainder term.

If the remainder is neglected, one has a linear equation which results in a mean square response.

$$\overline{y^2} = \frac{S_{ia}(f_n)}{4\alpha\lambda\ell^2} = \sigma_d\frac{\omega_o^2}{\lambda} \tag{9.45}$$

It will now be assumed the R_e can be neglected if the expectation $E\left[R_e^2\right]$ is minimized by a proper choice of λ. Lyon, Ref. [21], gives the proper choice as

$$\lambda = \omega_n^2(1 + 3\overline{y^2}) \tag{9.46}$$

Substituting Eq. (9.46) into Eq. (9.45) results in

$$\overline{y^2} = -\frac{1}{6} + \frac{\sqrt{(1 + 12\sigma_d^2)}}{6} \tag{9.47}$$

The ratio $\overline{y^2}/\sigma_d^2$ is plotted versus the linear mean square response σ_d^2 (which is a function of the input power spectrum) in Figure 9.9 and is also compared to the exact solution. As can be seen from Figure 9.9, the approximation is quite good resulting in errors of less than 10%.

Lyon, Ref. [21], also calculated the "average resonant frequency" using this approximation and the results are shown in Figure 9.10.

Figure 9.10 shows that even for small values of the input, the approximation is quite poor and it would be better to estimate the "average resonant frequency" of the nonlinear system by f_n.

Crandall, Ref. [9], has recently used a different approach in calculating $\bar{\nu}_0^+$ by approximate methods which agrees much more closely to the exact solution. He uses the method of equivalent linearization to obtain the free vibration-amplitude relation and then finds the average frequency of the random response by statistically averaging these frequencies using the probability density of the amplitudes.

9.2.4 Nonlinear System, Narrow-Band Excitation

One of the well known characteristics of nonlinear single degree-of-freedom systems subjected to sinusoidal excitation is the "Jump Phenomena." This is illustrated for the case of hardening spring in Figure 9.11.

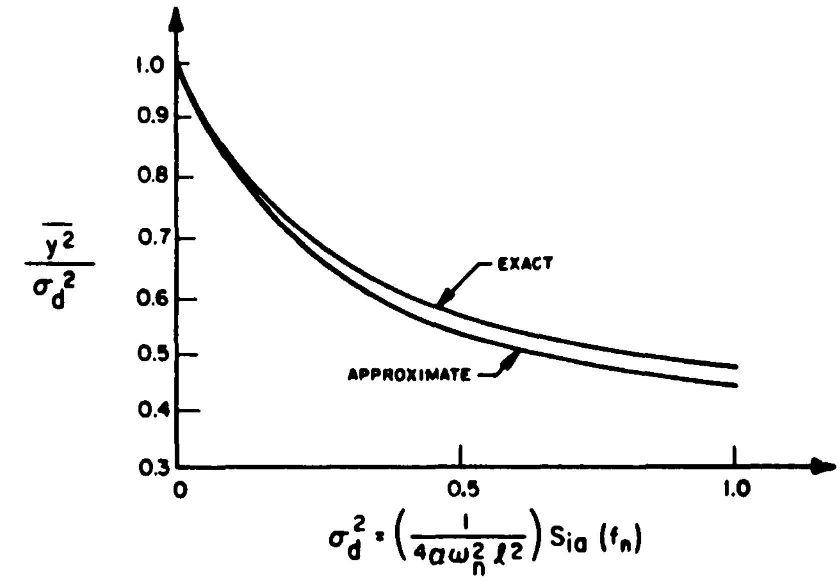


Figure 9.9. Comparison of Exact and Approximate Solution of the Mean-Square Response, Ref. [20] .

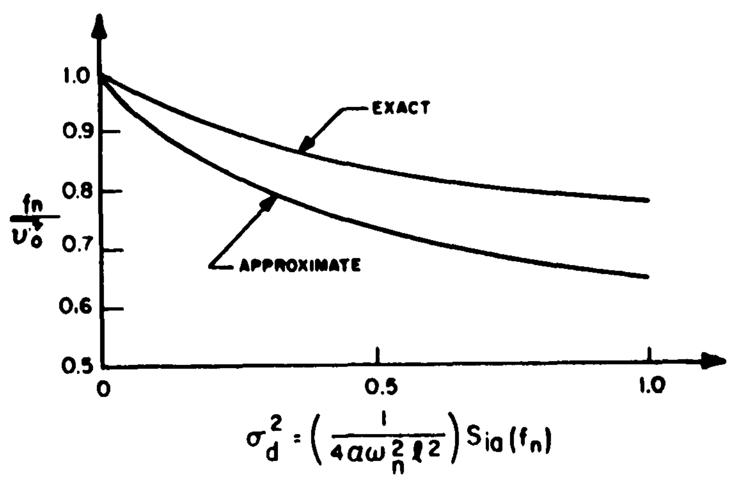


Figure 9.10. Comparison of Exact and Approximate Solution of the Ratio of Linear Natural Frequency to "Average Resonant Frequency"

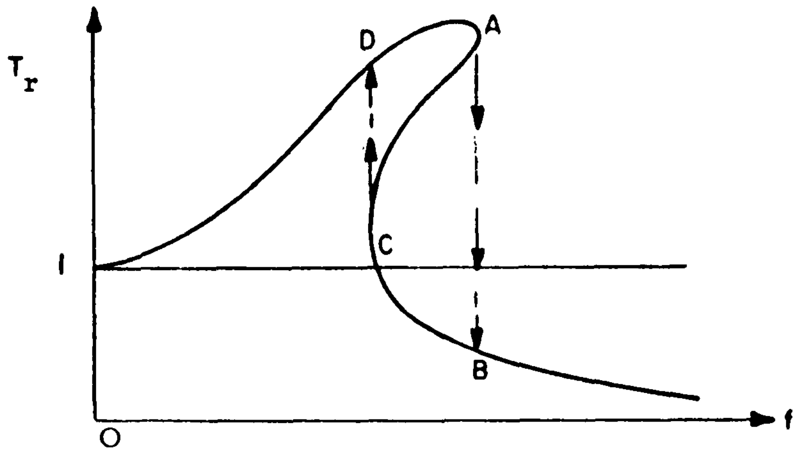


Figure 9.11. Response of Nonlinear, Damped, Single Degree-of-Freedom System to Sinusoidal Excitation.

As the frequency is slowly increased from zero cps, the transmissibility increases from 1 to D and to A and then "jumps" out of resonance to point B. If resonance is approached from the region above resonance by slowly decreasing the frequency, the transmissibility increases from B to C and then "jumps" into resonance to point D and then decreases to 1 at zero frequency. It should be noted that the region CA is unstable and therefore cannot represent the transmissibility of a physical system at any time.

Such a behavior does not seem to appear when the input is described by a wide-band random process and it can be demonstrated that such multi-valued behavior cannot exist when the excitation is wide-band. What is required is a source that can exchange energy with the system over several cycles in order that more than one stable state can exist consistent with the equations of motion. This argument suggests that a random input that correlates with itself and therefore

with the response over several cycles would produce the same effect. That this is so, has been shown by Lyon, Heckl, and Hazelgrove, Ref. [23], and a brief discussion of their method follows.

First, the magnitude response function of the nonlinear system is calculated by the use of Duffing's method, see Ref. [36]. Then it is assumed that the nonlinear system is driven with the output of a narrow band linear filter which in turn is driven by a wide band random process. The power spectrum of the response can then be calculated as follows:

$$G_o(\omega_n) = \left|H(\omega)_L\right|^2 \left|H(\omega)_N\right|^2 G_i(\omega_n) \tag{9.48}$$

where $\left|H(\omega)_L\right|$ is the magnitude response function of the linear narrow-band filter and $\left|H(\omega)_N\right|$ is the magnitude response function of the nonlinear system.

The results are plotted in Figure 9.12 and compared to the response curves for sinusoidal excitation. The symbols are defined as follows:

ω_n = linear natural frequency of single degree-of-freedom system

ω_f = center frequency of narrow-band filter

$B_{1/2}$ = half bandwidth of narrow-band filter

σ_L^2 = mean square value of output of filter (input to nonlinear system), normalized

$\overline{y^2}$ = mean square value of response, normalized

T_N^2 = $\overline{y^2}/\sigma_L^2$

The fraction of critical damping ζ in the system excited by sinusoidal excitation is .05 and in the system excited by narrow-band random excitation is "very small" (i.e., $\zeta \ll .05$) but not specified.

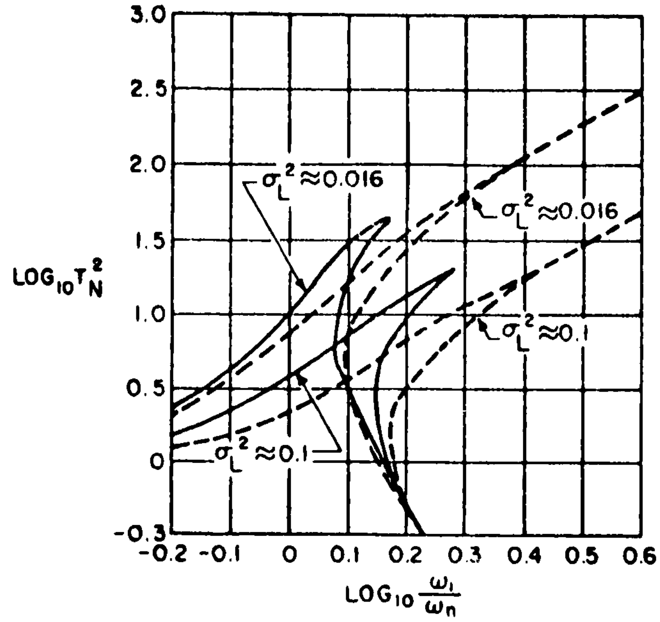


Figure 9.12. Comparison of Theoretical Response For Sinusoidal (solid) and Narrow-Band (dashed) Excitation for

$$\frac{a}{B_{1/2}} = 100 \text{ (filter very narrow)}$$

where $a = \omega_n \zeta$.

The similarity of the curves in Figure 9.12 to the curve in Figure 9.11 can be seen immediately. Their shapes differ somewhat mainly because Figure 9.12 is a log-log plot, whereas Figure 9.11 plots the relationship of T and f directly. However, an interesting question arises here, namely, how much time is spent in each of the three possible amplitude regions during random excitation. For the sinusoidal case it is known that the region CA (Figure 9.11) is unstable and the system can spend all of its time either along DA or CB, depending on how the operating point is approached. For the random input, the operating point probably changes constantly but how this change takes place, or how much time is spent at each operating point is not known at this time.

9.3 HARDENING SPRING — CONTINUOUS STRUCTURES

When considering the results of analyses for the response of continuous structures, one observation can be made immediately. Namely, even for sinusoidal excitation the analytical results are often poor approximations when compared to experimental data. This is due to the very limited knowledge about the mechanism of structural damping. It is known that for many structures the damping is of a nature which results in coupling of the various modes of vibration, i. e., the participation of the structure in one mode is dependent upon its participation in all other modes. The nature of this dependence is not known and unless simplifying assumptions are made, the equations of motion cannot be solved. The usual assumption made for structural damping is that it is in phase with the velocity and proportional to the generalized displacement of each mode.

9.3.1 General Theory

Before discussing nonlinear structures, a brief review will be given for the analysis of linear structures. Such a structure has certain normal mode properties and because of the orthogonality of these normal modes it can be assumed that the response of each individual mode is independent of the response in all the other modes.

The total displacement response, using the principal of superposition, is then simply the sum of the individual mode displacements. In terms of the normal modes, the total response is

$$u(x, y, z, t) = \sum_{n} q_n(t)\, \phi_n(x, y, z) \tag{9.49}$$

where $q_n(t)$ is the generalized displacement as a function of time and $\phi_n(x, y, z)$ the mode shape of the nth mode. To obtain $\phi_n(x, y, z)$ several approaches can be used. One of these is the Rayleigh-Ritz method, Ref. [15, p. 7-3], where a mode shape is assumed initially, and written in the form of a series with arbitrary constants. Solution of the Lagrange equations using this assumed mode shape will then result in values for these arbitrary constants that will give the best approximation to the actual mode shape.

9.3.2 Linear System

Making the assumption for structural damping as discussed above, let

$$\gamma_n = 2\zeta \frac{\omega}{\omega_n} \tag{9.50}$$

The Lagrange equations of the system will then have the form, see Ref. [15, p. 11-11]

$$\ddot{q}_n + \frac{\gamma_n}{\omega} \omega_n^2 \dot{q} + \omega_n^2 q_n = \frac{Q_n(t)}{M_n} \tag{9.51}$$

where

γ_n = structural damping coefficient

ω_n = nth normal angular frequency

M_n = $\int \phi_n^2(x, y, z)\, m(x, y, z)\, dx\, dy\, dz$ = generalized mass

$Q_n(t)$ = $\int F(x, y, z, t)\, \phi_n(x, y, z)\, dx\, dy\, dz$ = generalized input force

The generalized mass and input force can be obtained by integrating over the structure.

If a sinusoidal force is applied, one obtains

$$Q_n(t) = e^{j\omega t} \int F(x, y, z)\, \phi_n(x, y, z)\, dx\, dy\, dz = e^{j\omega t} w_o w_n \tag{9.52}$$

where

$$w_o = \int F(x, y, z)\, dx\, dy\, dz = \text{amplitude of total applied force}$$

and

$$w_n = \frac{1}{w_o} \int F(x, y, z)\, \phi_n(x, y, z)\, dx\, dy\, dz = \text{mode participation of the applied force}$$

The steady-state solution for Eq.(9.51) is straightforward, resulting in

$$q_n = \frac{w_o w_n e^{j\omega t} e^{j\Theta_n}}{M_n \omega_n^{\,2} \sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \gamma_n^{\,2}}} \tag{9.53}$$

Substituting into Eq.(9.49) results in a total displacement of the structure

$$u = w_o e^{j\omega t} \sum_n \frac{w_n \phi_n(x,y,z)\, e^{j\Theta_n}}{M_n \omega_n^{\,2} \sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \gamma_n^{\,2}}} \tag{9.54}$$

The square of the magnitude response function, defined here as the ratio of output displacement to input force can now be determined from Eq.. (9.54), and one obtains

$$\left|A(\omega)\right|^2 = \left|\frac{u}{w_o}\right|^2 = \sum_n \frac{w_n^2 \phi_n^{\,2}(x, y, z)}{M_n^2 \omega_n^{\,4} \left\{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \gamma_n^{\,2}\right\}} \tag{9.55}$$

The solution for the response to random excitation can now also be found very easily by the use of Eq. (9.16). This results in a mean square displacement response

$$\overline{u^2} = \frac{\pi}{2} \sum_n \frac{w_n^{\,2} \phi_n^{\,2}(x, y, z)\, G_{iF}(\omega_n)}{\gamma_n M_n^{\,2} \omega_n^{\,3}} \tag{9.56}$$

Eq.(9.56) will reduce to Eq.(9.20) if only the first mode is assumed to participate in the response and the structure is considered as a lumped mass with an equivalent spring constant k. Since, then, $w_n^2 \phi_n^2 (x, y, z) = 1$, $M_n^2 = m^2 = k^2/\omega_n^4$, $\gamma_n = 2\zeta \frac{\omega}{\omega_n}$, and $\frac{\omega}{\omega_n} \approx 1$

9.3.3 Nonlinear System

As has been shown above, a set of independent uncoupled modes can be used to describe the response of linear structures. This is not the case for nonlinear structures. For example, consider the case of a pinned-pinned beam.

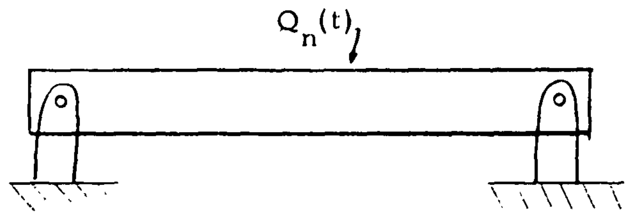


Figure 9.13. Pinned-Pinned Beam

First, Eq (9.51) can be rewritten as follows:

$$M_n \ddot{q} + R_n \dot{q} + K_n q = Q_n(t) \tag{9.57}$$

where R_n is the generalized damping resistance and K_n the generalized stiffness. The particular Lagrange equation for the pinned beam can be derived from the kinetic and potential energies just as in the linear case. However, the generalized stiffness K_n derived from the potential energy will now exhibit some major differences. The stiffness for the ith mode will now have the following form, see Ref. [34, p. 5]

$$K_i = K_{io} \left[1 + \frac{\sum_n n^2 q_n^2}{i^2 \ell^2} \right] \tag{9.58}$$

where l is the characteristic length equal to twice the radius of gyration of the beam's section. This clearly indicates the coupling of the modes and that the response of one mode will require consideration of the response in all other modes.

The damping term R_n also adds to this problem. As for the linear case, insufficient knowledge exists as to the exact nature of the damping mechanism. In addition, in the nonlinear case, even if the damping mechanism is linear, the damping resistance R_n is nonlinear, see Ref. [35]. The result of these difficulties is that the equations of motion for an exact, or even almost exact, mathematical model for a damped nonlinear structure have not been solved to date. Therefore, most analyses made in recent years usually assumed 1) linear damping, and 2) response only in the first mode using a lumped mass and equivalent spring constant. This type of system has been discussed in Sections 9.2.3 and 9.2.4. However, some experimental results are available and will be discussed below.

Lyon recently published data from an experimental study of a clamped-clamped beam subjected to narrow-band random excitation, Ref. [23] (also, see Section 9.2.4 for theoretical discussion). The fundamental resonance was placed at 75 cps, the bandwidth at resonance was 10.5 cps, and the Q of the beam was 7.15. The output of the random noise generator was filtered using an 8% bandwidth setting. The experimental results are shown in Figure 9.14(a), (b), and (c), and are compared to theoretical results based on the single degree-of-freedom model. It should be noted that the damping for the theoretical curves was small (i.e., $\zeta \ll .05$) However, all other parameters were the same as those for the experimental model.

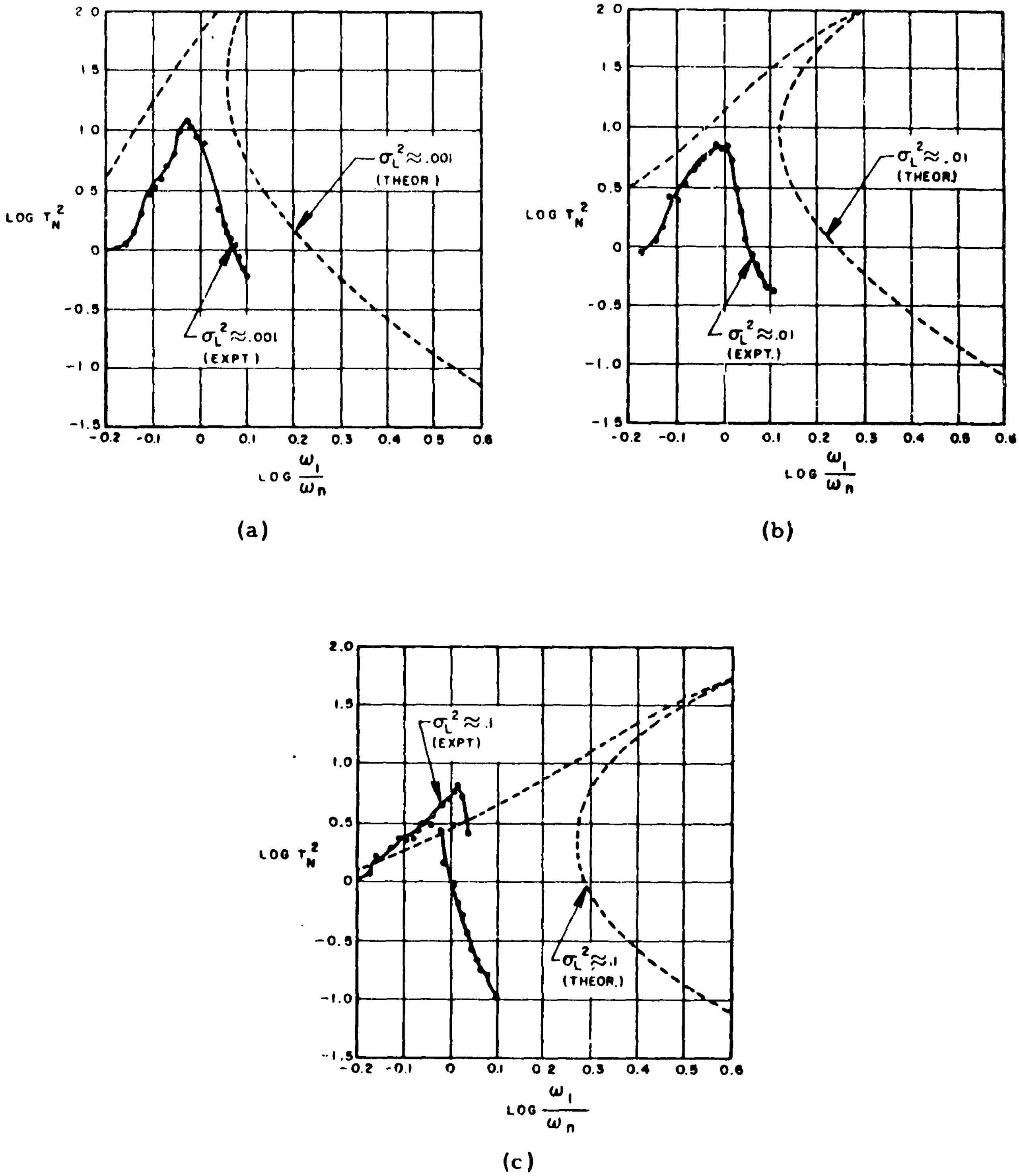


Figure 9.14. Comparison of Experimental Response Amplitude with Theoretical Calculations (Lyon, Ref. 23)

The agreement with theory, as indicated by Figure 9.14, is not good. Particularly the jump phenomena (Figure 9.14(c)) occurs at a much lower frequency than that predicted by theory. This may be due to the large difference in damping between the two cases or other factors, such as nonlinear damping, and/or higher mode participation.

9.3.4 Fatigue

Another topic of interest in the analysis of response of structures is the fatigue characteristics of the structure. There are three additional parameters which now have to be considered. These are the probability density of peaks, the displacement-strain relationship, and the probability density of stress maxima. In order to determine these parameters it was assumed that the response occurred only in the first mode and the beam was replaced by an equivalent spring constant and a lumped mass. Therefore, the accuracy of the results should be eyed with suspicion. The probability density of peaks is given by Ref. [20]:

$$p(y) = \frac{y(1+y^2)}{\sigma_d^2} \exp\left[\frac{-y^2\left(1+\frac{y^2}{2}\right)}{2\sigma_d^2}\right] \tag{9.59}$$

remembering that y and σ_d are normalized with respect to the characteristic length ℓ (i.e., $y = \frac{z}{\ell}$ and $\sigma_d = \frac{\sigma_r}{\ell}$). It now is convenient to normalize the characteristic length ℓ and the displacement z with respect to the root-mean-square response σ_r. Defining $z_n = \frac{z}{\sigma_r}$ and $\ell_n = \frac{2\ell}{\sigma_r}$, Eq. (9.59) becomes, Ref. [32],

$$p(z_n) = \frac{z_n \ell_n}{2}\left(1 + 4\frac{z_n^2}{\ell_n^2}\right)\exp\left[-z_n^2\left(\frac{1}{2} + \frac{z_n^2}{\ell_n^2}\right)\right] \tag{9.60}$$

The probability density of peaks $p(z_n)$ for $\ell_n = 8$ is shown in Figure 9.15, obtained from Ref. [34].

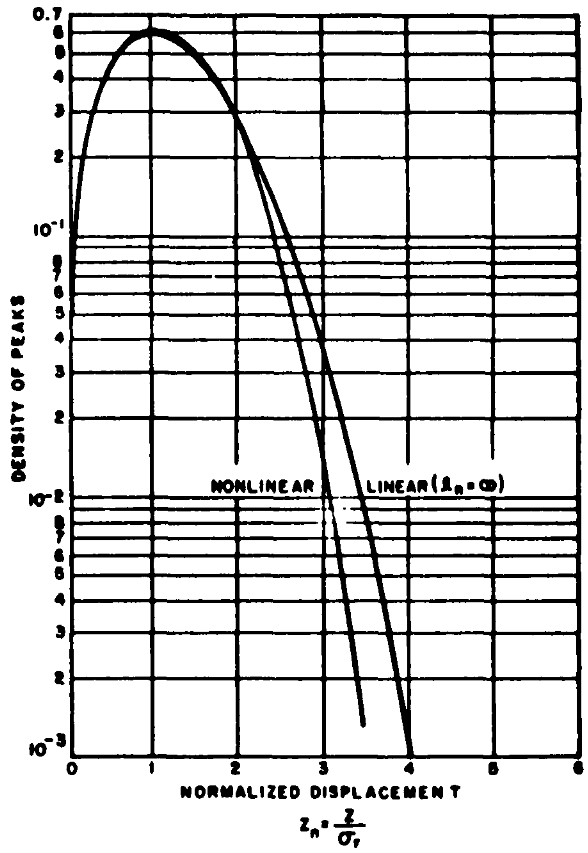


Figure 9.15. Density of Peaks of a Linear and Nonlinear Single Degree-of-Freedom System for $\ell_n = 8$.

As discussed earlier, Figure 9.15 also indicates that the probability of high peaks occurring with a relatively large input is less for the non-linear than for the linear system.

The following assumptions are made in Ref. [32]. Fatigue damage is cumulative and the fractional damage contributed by any one response peak is a function of the peak amplitude of strain only. The damage contributed by a single peak of strain ϵ_s is proportional to $(\epsilon_s)^\beta$ where β is the negative of the slope of the log ϵ_s vs. log N curve obtained from fatigue tests made with constant amplitude peaks.

Two relationships are now assumed between the strain ϵ_s and the response displacement z. First, a linear relationship, $\epsilon_s = c_1 z$, second, a nonlinear relationship, $\epsilon_{st} = c_2 z(1 + z/2\ell\sqrt{3})$ where c_1 and c_2 are constants. Of course, other relationships also exist, each depending on the particular structure in question.

The strain ϵ_{st} for instance, is the total strain in the surface fibers at the center of a rectangular pinned-pinned beam, vibrating at its fundamental mode and is due to a bending component ϵ_{sb} proportional to z and a membrane component ϵ_{sm} proportional to z^2.

For the first relationship the damage rate, or fractional damage per cycle is proportional to, see Ref. [32]

$$R_A = \int_0^\infty (z_n)^\beta \, p(z_n)\, dz_n \qquad (9.61)$$

Since ϵ_s is proportional to z, z_n can be interpreted as the strain normalized to σ_r, the linear rms response.

The estimate of the damage rate for a linear system is given by Eq. (9.62) for comparison

$$R_0 = \int_0^\infty (z_n)^{\beta+1} e^{-z_n^2/2} \, dz_n = 2^{\beta/2}\Gamma\left(\frac{1}{2}\beta + 1\right) \qquad (9.62)$$

For the second relationship, the normalized strain is given by

$$\epsilon_{stn} = z_n (1 + z_n/\ell_n\sqrt{3}) \qquad (9.63)$$

Here the damage rate is proportional to

$$R_B = \int_0^\infty \left[\epsilon_{stn}(z_n)\right]^\beta p(z_n)\, dz_n \qquad (9.64)$$

The integrands for Eqs. (9.61), (9.62), and (9.64) can be called "Damage Densities" and are plotted for $\beta = 8$ and $\ell_n = 8$ in Figure 9.16, Ref. [32].

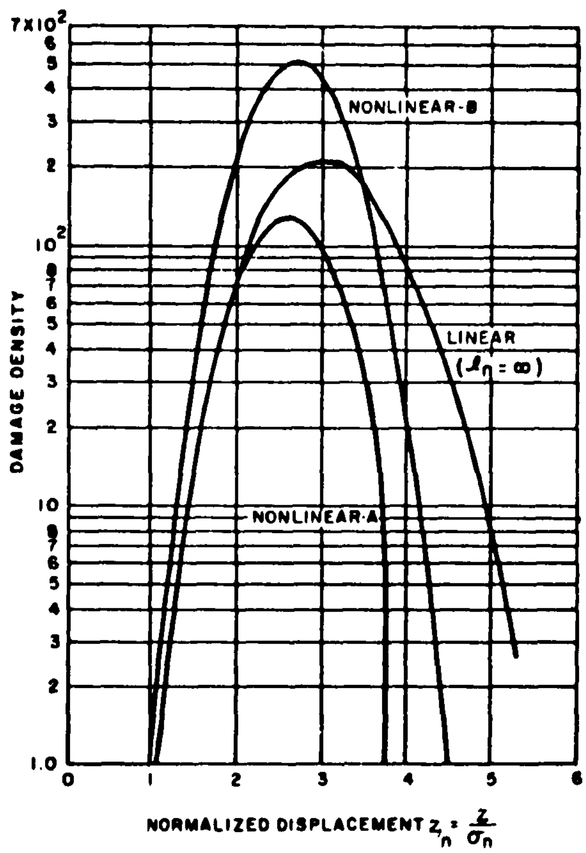


Figure 9.16. Damage Density vs. Normalized Displacement Response

Figure 9.16 indicates that for large values of z_n the density of peaks is considerably less for the nonlinear system than for the linear case which agrees with the previous discussion in Section 9.2.3. Similarly for both cases of the strain relationships the probability of damage for

large values of z_n is less for the nonlinear system than for the linear system for given values z_n.

However, in addition, one should also consider the probability of exceeding some arbitrary level of stress. In the previous discussion it was shown that the probability of high amplitudes was reduced for the nonlinear system. However, one can almost predict intuitively that at some given large displacement, the stress in the nonlinear system will be higher than in the linear system. Lyon, Ref. [22] made an approximate analysis for determining the probability density of stress maxima and minima for a simply supported bar excited in the first mode by random noise and assuming the hardening spring law. The qualitative results are shown in Figures 9.17 and 9.18.

Figure 9.17 shows clearly in the expanded scale that the probability of large stresses occurring is much higher for the nonlinear system than for the linear system. A numerical example worked out by Lyon shows that the fatigue life could be cut by as much as a factor of 2.

Recent experiments by D.A. Smith and R. F. Lambert, Ref. [31], have resulted in qualitative agreement with this prediction.

Experiments conducted with cantilever beams, Ref. [14], also resulted in an actual fatigue life which was shorter by as much as a factor of 2 when compared to that predicted by linear theory when subjected to random vibration. However, an explanation for this result was not given and it may or may not have been due to nonlinearities.

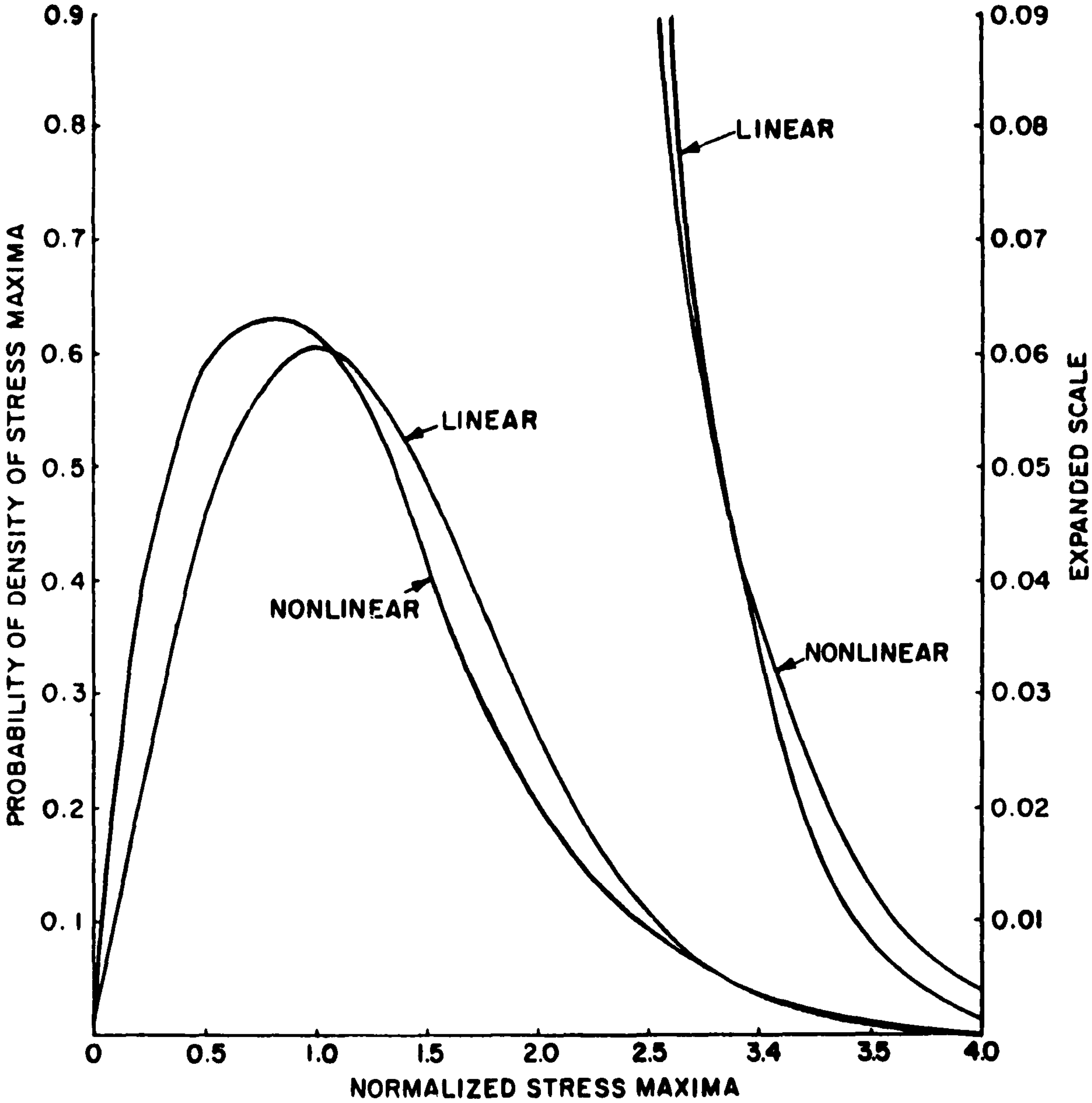


Figure 9.17. Density of Stress Maxima

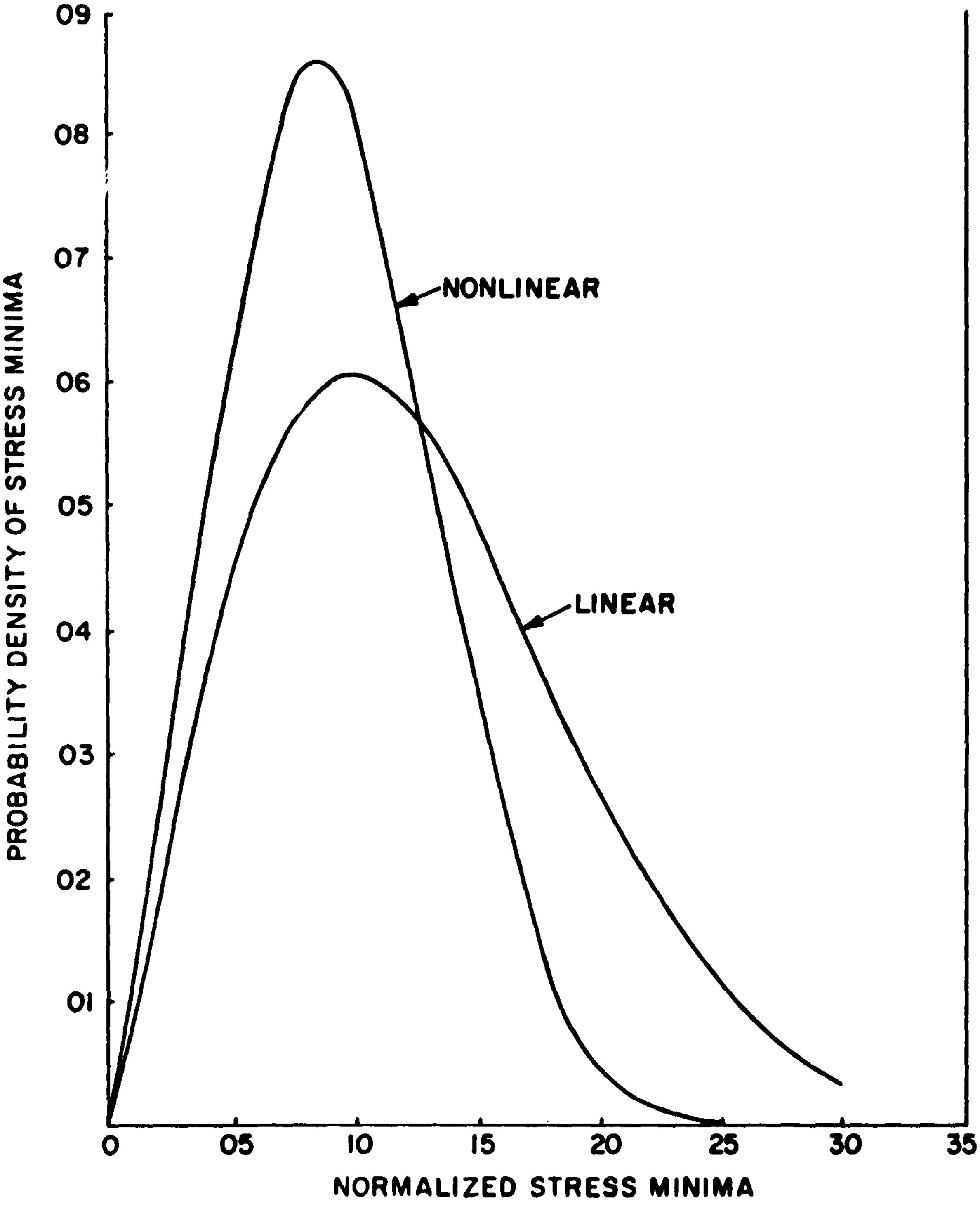


Figure 9.18. Density of Stress Minima

9.4 OTHER TYPES OF NONLINEARITIES

9.4.1. Set-Up Spring

Recently Crandall, Ref. [10], made an analysis determining the response of a nonlinear system with a set-up spring. The load deflection characteristics of this system is shown in Figure 9.19.

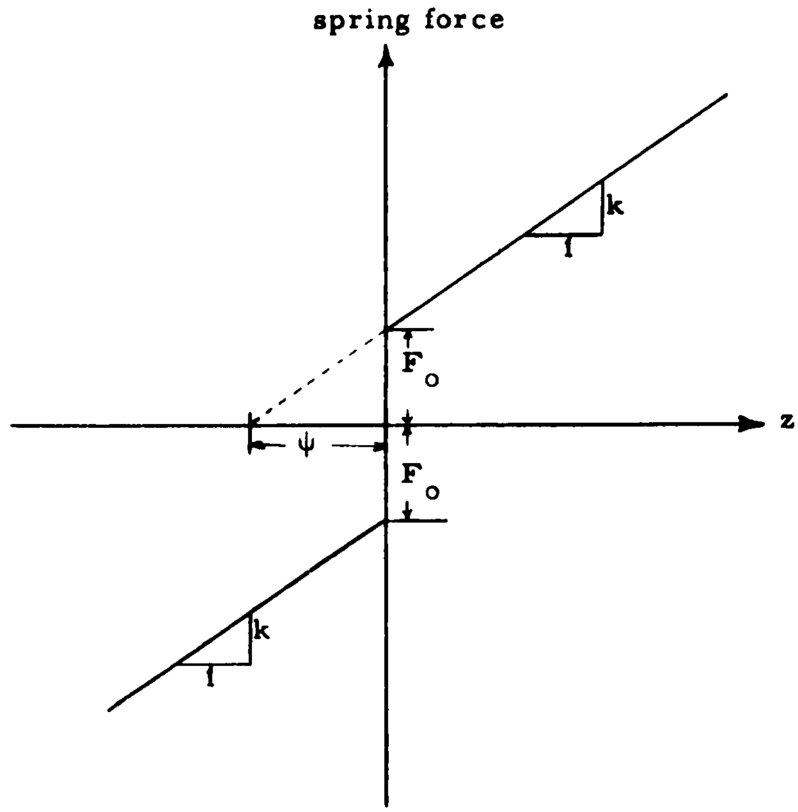


Figure 9.19. Nonlinear Load-Deflection Curve of Set-Up Spring

The equation describing this characteristic is

$$F(z) = k(z + \psi \operatorname{sgn} z) \tag{9.65}$$

where $\psi = F_o/k$.

Letting $k = m\omega_n^2$ one can write

$$\frac{1}{m} F(z) = \omega_n^2 (z + \psi \operatorname{sgn} z) \qquad (9.66)$$

Substituting Eq. (9.66) into Eq. (9.26) one obtains

$$\ddot{z} + 2a\,\dot{z} + \omega_n^2 (z + \psi \operatorname{sgn} z) = A(t) \qquad (9.67)$$

It should be noted from Fig. 9.19 that when the mass m passes through $z=0$ a jump occurs in the relative acceleration of magnitude $2F_o/m$ but that its relative velocity is continuous. If initially z and $\dot{z}$ are zero, they will remain zero until the magnitude of $A(t)$ exceeds the value F_o/m. It will be assumed that $A(t)$ is large enough so that the possibility of z and $\dot{z}$ being zero occurs very rarely. The spectral density of $A(t)$ is defined in units of acceleration squared per radian/sec over both positive and negative frequencies with $\bar{\ddot{z}} = 0$

Solving Eq (9.67) exactly by the application of the Fokker-Planck equation (as discussed in Section 9.2.3) results in the following probability densities for z and $\dot{z}$, Ref. [10].

$$p(z) = \frac{\exp - \left[\dfrac{\psi^2}{2\sigma_r'^2}\right]}{\sqrt{2\pi}\ \sigma_r' \operatorname{erfc}(\psi/\sigma_r'\sqrt{2})} \exp\left[\frac{-\left(z^2 + 2\psi z \operatorname{sgn} z\right)}{2\sigma_r'^2}\right] \qquad (9.68)$$

and

$$p(\dot{z}) = \frac{1}{\sqrt{2\pi}\ \omega_n \sigma_r'} \exp\left[\frac{-\dot{z}^2}{2\omega_n^2 \sigma_r'^2}\right] \qquad (9.69)$$

where $\sigma_r'^2 = 2\pi\sigma_r^2$ [see Eq.(9.33)] is the mean square response if the system were linear.

The "average resonant frequency" is

$$\bar{\nu}_0^+ = \frac{\omega_n \exp\left[-\psi^2/2\sigma_r'^2\right]}{2\pi \operatorname{erfc}(\psi/\sigma_r'\sqrt{2})} \tag{9.70}$$

The mean square displacement from Eq. (9.68) becomes

$$\overline{z^2} = \int_{-\infty}^{\infty} z^2\, p(x)\, dx = \sigma_r'^2 \left[1 - \frac{\psi}{\sigma_r'}\left(\frac{2}{\pi}\right)^{1/2} \cdot \frac{\exp\left[-\psi^2/2\sigma_r'^2\right]}{\operatorname{erfc}(\psi/\sigma_r'\sqrt{2})} + \frac{\psi^2}{\sigma_r'^2}\right] \tag{9.71}$$

An approximate solution to Eq.(9.67) can be obtained by the application of the "equivalent linearization" technique previously discussed in Section 9.1.1.3.

Similar to Eq.(9.43) one can write

$$\ddot{z} + 2\alpha\dot{z} + \lambda z = A(t) + R_e \tag{9.72}$$

where the remainder term R_e now is

$$R_e = \lambda z - \omega_n^2(z + \psi \operatorname{sgn} z) \tag{9.73}$$

Solving Eq.(9.72) as discussed in Section 9.2.3.(b) by minimizing $E(R_e^2)$, one obtains, Ref. [10].

$$\overline{z^2} = \sigma_r'^2 \left[1 + \left(\frac{2}{\pi}\right)^{1/2} \frac{\psi}{\sigma_r'}\right]^{-1} \tag{9.74}$$

and

$$\bar{\nu}_0^+ = \frac{\omega_n}{2\pi}\left[1 + \left(\frac{2}{\pi}\right)^{1/2} \frac{\psi}{\sigma_r'}\right] \tag{9.75}$$

Figure 9.20 shows the probability density p(z) obtained from the exact solution for various values of the nonlinearity parameter $\psi/\sigma_r' \sqrt{2}$. Figure 9.21 plots the ratio of nonlinear to linear mean-square response vs. the nonlinearity parameter and compares the approximate and exact solutions. Figure 9.22 plots the ratio of nonlinear "equivalent resonant frequency" to linear natural frequency f_n vs. the nonlinearity parameter for both the exact and approximate solutions.

For fatigue studies the probability density of peaks $p(z_m)$ is of interest. Using the result of Eq. (9.68), Crandall, Ref. [10] shows this to be

$$p(z_m) = \frac{(z_m + \psi)}{\sigma_r'^2} \exp\left[\frac{-(z_m^2 + 2\psi z_m)}{2\sigma_r'^2}\right] \tag{9.76}$$

The probability density of the envelope connecting the peaks is given by

$$p(z_m) = \frac{(z_m + \psi)}{\sigma_r'^2} \exp\left[\frac{-(z_m^2 + 2\psi z_m)}{\sigma_r'^2}\right] \frac{\exp\left[-\psi^2/2\sigma_r'^2\right]}{\operatorname{erfc}(\psi/\sigma_r'\sqrt{2})} \frac{2}{\pi} \cos^{-1}\left(\frac{\psi}{z_m + \psi}\right) \tag{9.77}$$

Equations (9.76) and (9.77) are plotted in Figures 9.23 and 9.24 respectively for various values of the nonlinearity parameter $\psi/\sigma_r'\sqrt{2}$.

It should be noted that the distribution of Eq. (9.76) relates to the relative frequency of peaks of z_m in the population of all peaks whereas Eq. (9.77) relates to a smooth continuous envelope $z_m(t)$ which joins succeeding peaks.

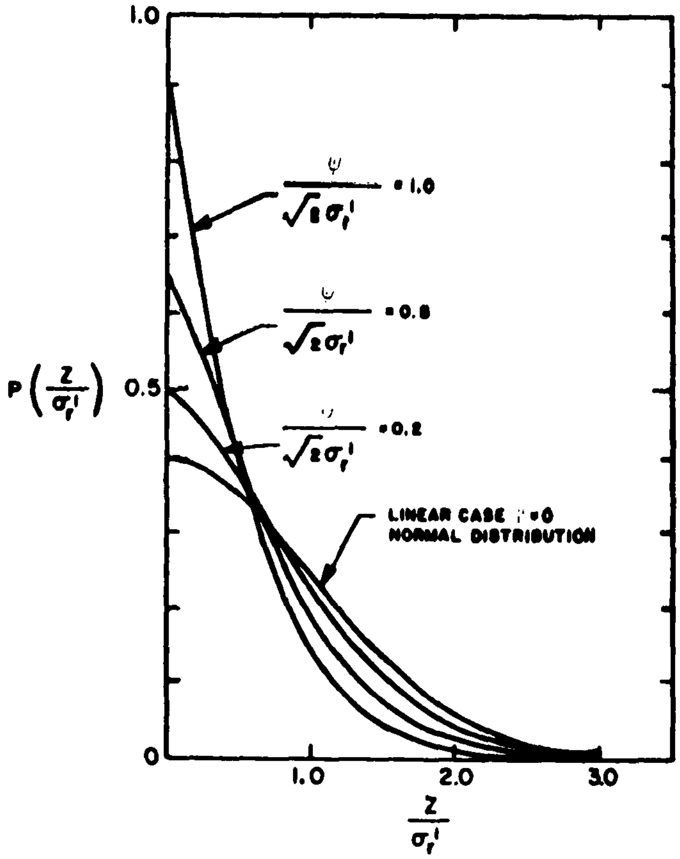


Figure 9.20. Probability Density vs. Normalized Displacement $\frac{Z}{\sigma_r'}$

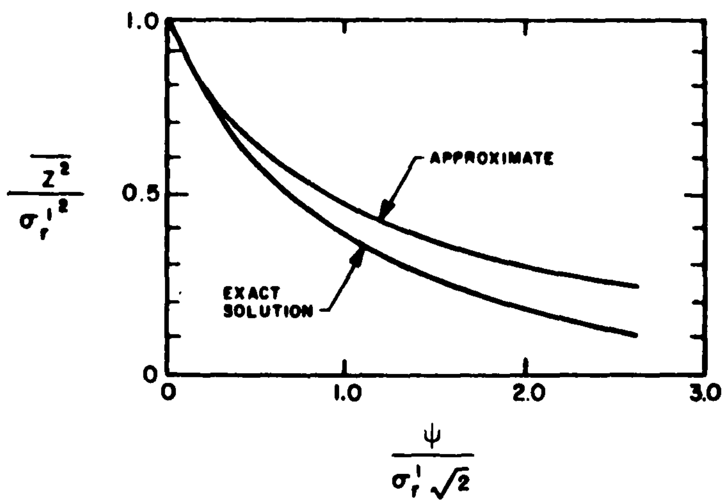


Figure 9.21. Ratio of Nonlinear to Linear Mean-Square Response vs. Nonlinearity Parameter $\psi/\sigma_r'\sqrt{2}$

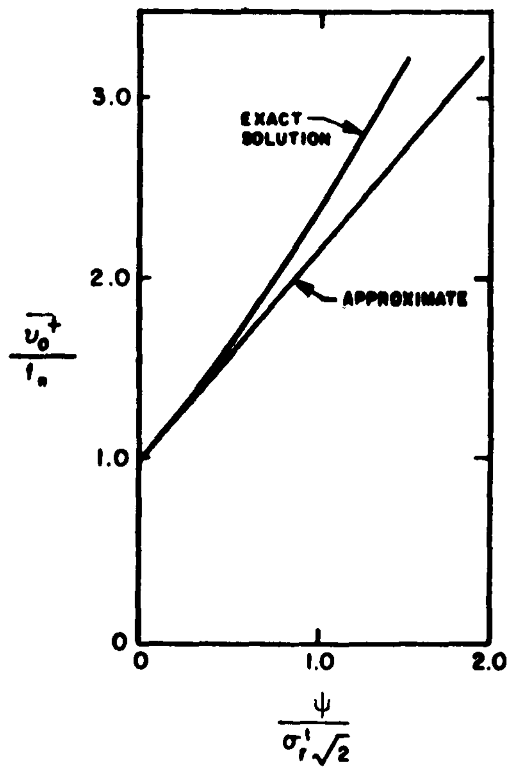


Figure 9.22. Ratio of Nonlinear "Average Frequency to Linear Natural Frequency vs. Nonlinearity Parameter $\psi/\sigma_r'\sqrt{2}$

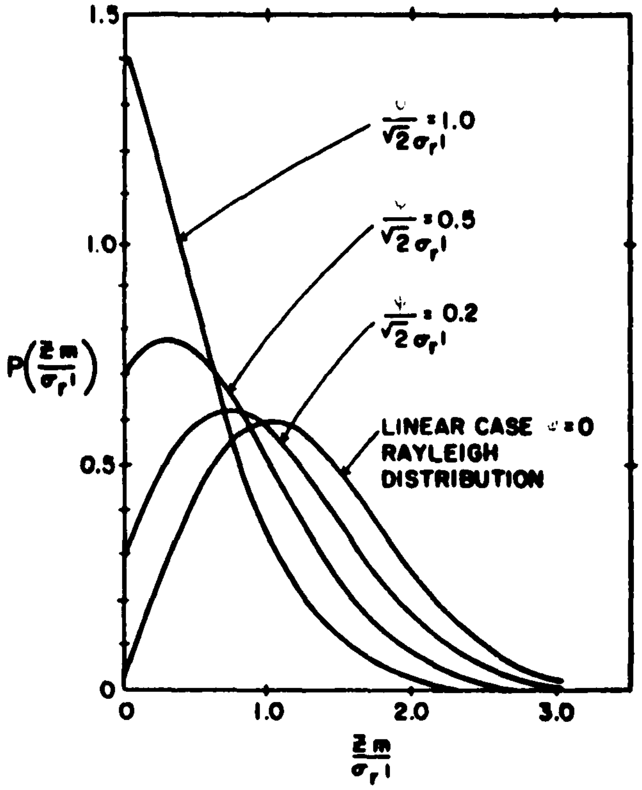


Figure 9.23. Probability Density of Peak Amplitudes for Equation (9. 76)

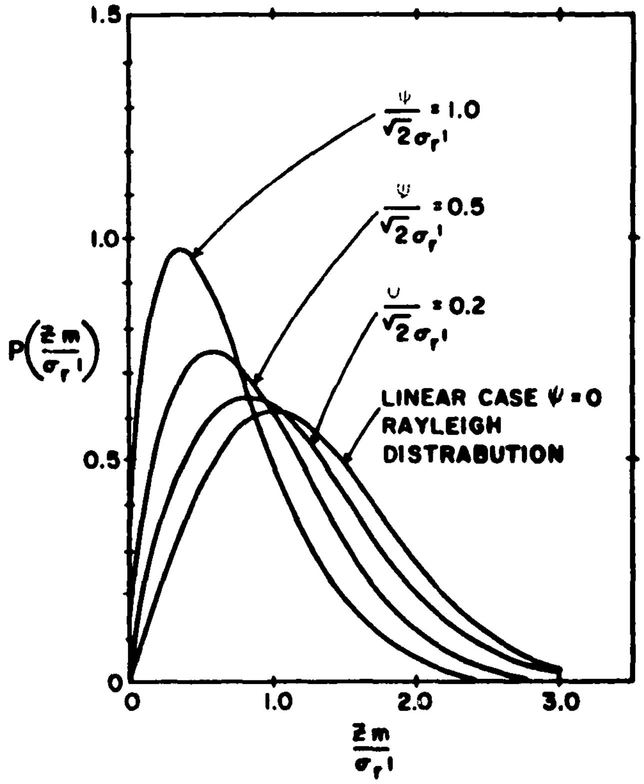


Figure 9.24. Probability Density for Envelope of Peaks $z_m(t)$ for Equation (9. 77)

9.4.2 Dead-Zone Spring

In a report written by R. E. Oliver and T. Y. Wu, Ref. [29], Part II, under contract No. AF 29(600)-1338, sponsored by the Air Force Missile Development Center, Holloman Air Force Base, New Mexico, the problem of random excitation of a dead-zone spring is solved. The general load deflection curve for this spring is shown in Figure 9.25.

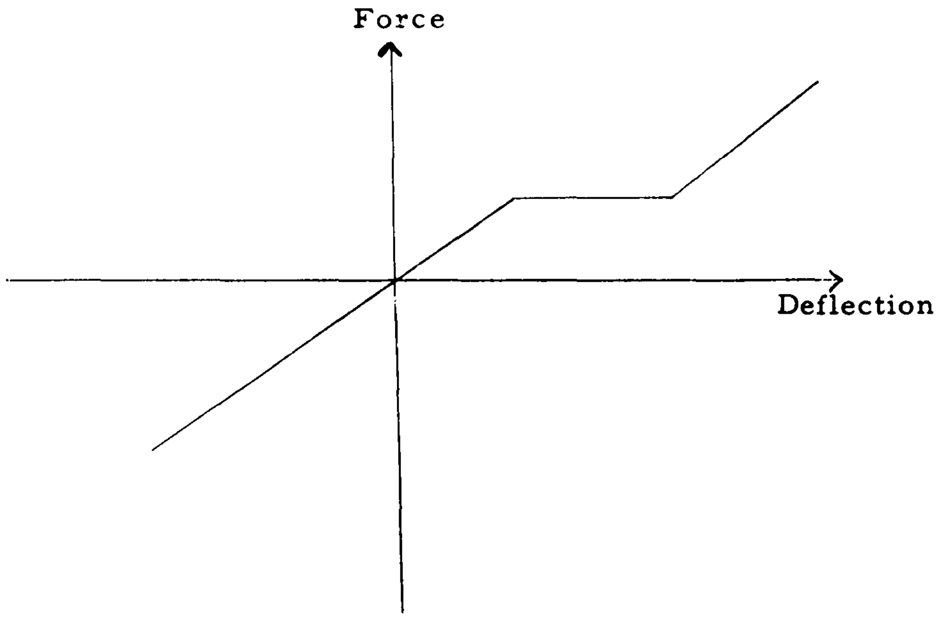


Figure 9.25. Load-Deflection Curve of Dead-Zone Spring

The analysis is quite extensive and it is therefore beyond the scope of this report to go into the detailed solution of this problem, and the reader is referred to Ref. [29].

9.4.3 Nonlinear Damping

Two types of nonlinear damping (combined with linear stiffness) are most often assumed for various physical applications. The first is combined linear and displacement-cubed damping sometimes referred to as hysteresis or material damping. The equation of motion would then be from Eq. (9.26)

$$\ddot{z} + 2a\dot{z} \pm c_1 z^3 + \omega_n^{\;2} z = A(t) \qquad (9.78)$$

where c_1 is a constant of proportionality and $c_1 z^3$ is positive when $\dot{z}$ is positive and $c_1 z^3$ is negative when z is negative.

The second considers combined linear and velocity-squared damping often referred to as air damping.

The equation of motion is

$$\ddot{z} + 2a\dot{z} + c_2 |\dot{z}| \dot{z} + \omega_n^{\;2} z = A(t) \qquad (9.79)$$

where c_2 is also a constant.

As far as is known, exact solutions to Eqs. (9.78) and (9.79), when A(t) is a random process, do not exist. McIntosh, Ref. [26], however, made an analog computer study resulting in approximate solutions for both equations for the root-mean-square acceleration response. Caughey, Ref. [6], obtained an approximate sdution for the special case of Eq(9.78) where the hysteresis loop is bilinear through the application of "equivalent linearization." The reader is referred to these two references for a detailed discussion since it is beyond the scope of this summary to do so here.

9.4.4 Instrumentation Nonlinearities

In addition to the considerations of the response of nonlinear mechanical systems to random excitation, is the response of instrumentation used for measuring and analyzing random phenomena. The following discussion is taken from Reference [2].

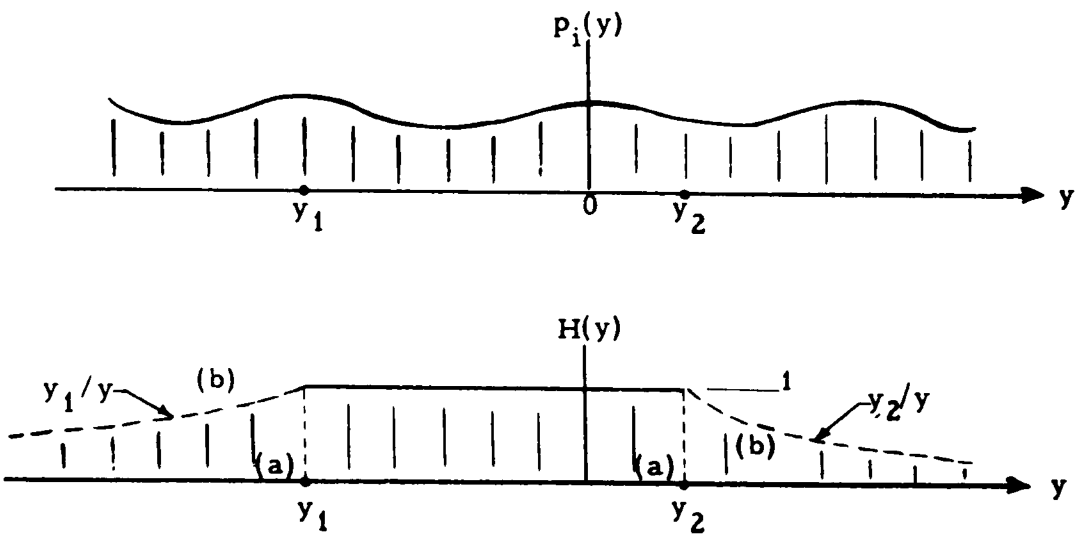
Consider first the case of nonlinear amplitude transfer characteristics (nonlinear gain) which might be associated with transducers and data processing instrumentation. Included in this category is the problem of limited dynamic range. When a random signal is passed through an amplitude transfer characteristic which displays amplitude linearity, the probability density function of the signal will not be changed. If, however, an amplitude transfer characteristic is not linear, the probability density function of the signal will be altered by the transfer characteristic. If the probability density function of the signal is altered, the mean value and the mean square value (and other statistics) of the signal will probably be changed also. The effect of nonlinear amplitude transfer characteristics on random signals is briefly discussed with illustrations in Section 9, Ref. [2]. A related general treatment of the effects of nonlinear transfer characteristics on the statistics of random signals is available from Reference [12], Chapters 12 and 13.

The term "limited dynamic range" might be interpreted in two ways The first interpretation is a transfer characteristic (gain function) with a gain of one for amplitudes with absolute values below some specified level, and a gain of zero for amplitudes greater than that specified level. In other words, an instantaneous input

amplitude with a value outside the dynamic range limits would result in an instantaneous output amplitude of zero. Such an interpretation of limited dynamic range has little physical significance. In reality, limited dynamic range usually implies a transfer characteristic which limits amplitudes with absolute values above some specified level to that specified level. This second interpretation is more representative of such phenomena as clipping, magnetic saturation, and similar physical limitations on amplitude linearity.

The specific effect of both interpretations of limited dynamic range on a random signal with a uniform input probability density function will now be developed.

Consider the general case of a stationary random signal with an amplitude probability density function $p_i(y)$ when passed through an amplitude transfer characteristic $H(y)$ with limited dynamic range.



Hypothetically, two interpretations of limited dynamic range are possible. The amplitude transfer characteristic $H(y)$ may be such that amplitudes of y less than y_1 or greater than y_2 are:

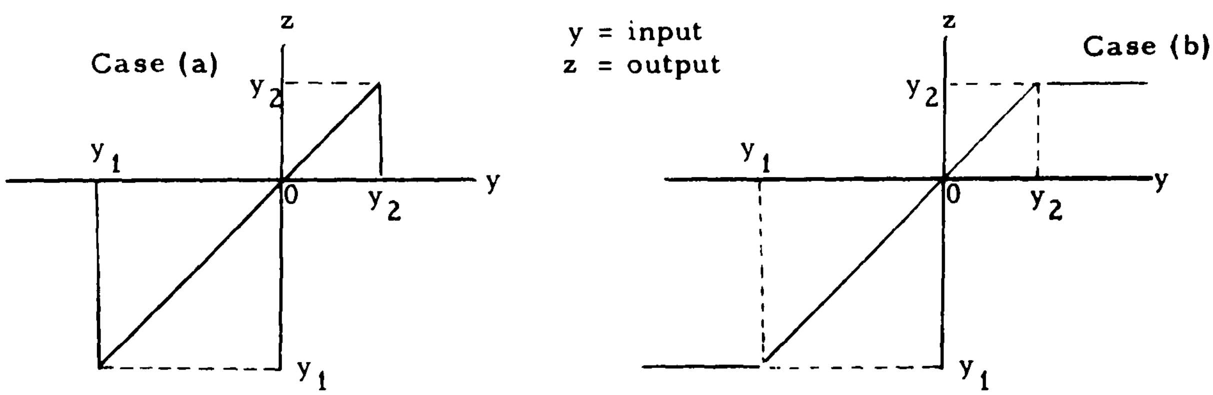
(a) excluded (equal to zero)

(b) limited to values of y_1 or y_2

Case (a) would occur in a device which "open-circuits" when input signals are above a level y_2 or below a level y_1.

Case (b) would occur in a device which clips or limits when input signals are above a level y_2 or below a level y_1.

The transfer characteristic $H(y)$ associated with each of the two interpretations for limited dynamic range may be thought of graphically as shown above. The vertical dashed lines labeled (a) complete the transfer characteristic for case (a) and the dashed lines labeled (b) complete the transfer characteristic for case (b). The two transfer characteristics may also be presented in the form of input-output plots as shown below.



If the input has a uniform probability density function, Reference [2] has shown that the output probability density will have the characteristics shown in Figure 9.26 for cases (a) and (b) discussed above.

Since for case (a) all amplitudes larger than y are zero, one would intuitively expect a clustering at $z = 0$. For case (b) all amplitudes larger than y are equal to $z = |b|$ and one would expect clustering at $z = |b|$. The theoretical results substantiate these expectations resulting in delta functions at $z = 0$ for case (a) and at $z = |b|$ for case (b).

INPUT: A uniform probability density function with a mean value of zero.

AMPLITUDE TRANSFER CHARACTERISTIC: A gain of one with limited dynamic range of two types as shown.

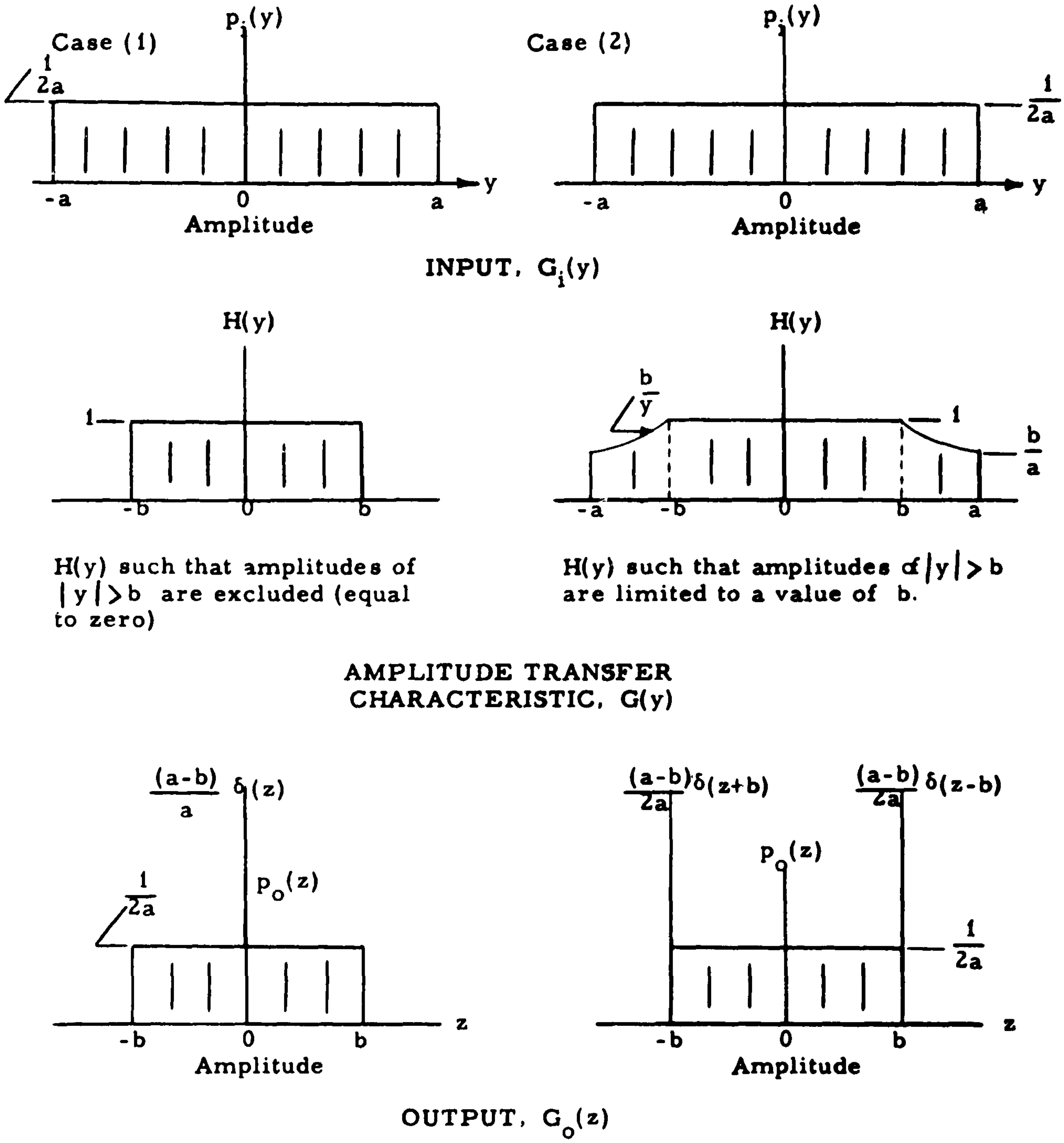


Figure 9.26. Output Probability Density Function as Function of Two Different Nonlinear Transfer Characteristics (Uniform Input Probability Density Function)

Pastel, Ref. [30], has given the output probability density when the input is Gaussian for the case No. 2 discussed above. This is shown in Figure 9.27(A). As in Figure 9.26 (case 2), delta functions appear at the tails of the output distribution. In addition, various output distributions are shown for several other types of transfer functions for a Gaussian input probability density function.

Of the above types of nonlinearities, limited dynamic range as represented by Figure 9.27(A) constitutes the most common problem in vibration measurement work. Even with the highest quality instrumentation, a test engineer may unduly limit the dynamic range of random signal measurements by improper use of the measurement equipment. A common example is the improper use of instruments which were designed primarily for harmonic signal work.

Many instruments used in association with vibration measurement and analysis are equipped with input attenuators and signal level meters which permit input signals levels to be appropriately adjusted for the dynamic range limits of the particular instrument. To keep the signal to noise ratio as high as possible, it is not uncommon for the test engineer to adjust the input attenuators so that signal levels are near the upper limits of the instrument's dynamic range. If the input signal level meter is a conventional average sensing voltmeter calibrated in rms for sine waves, the maximum input level to the instrument as defined on the meter would probably correspond to that voltage level where the instrument begins to clip sine waves. In other words, the instrument would clip voltage levels

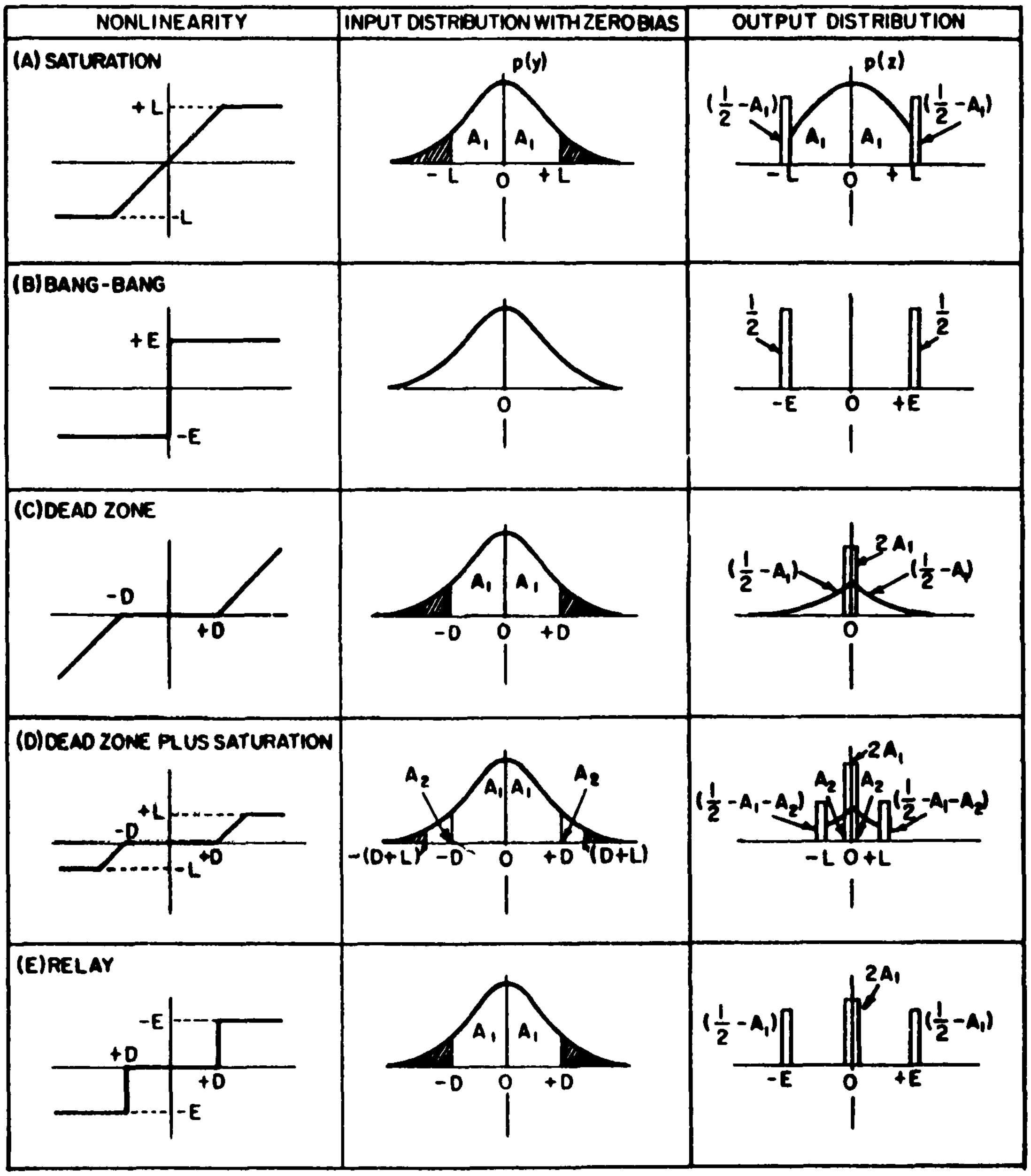


Figure 9.27. Output Probability Density Functions for Various Nonlinear Transfer Functions for a Gaussian Input

of about 1.4 times the maximum rms voltage input as defined on the meter.

For the case of input signals which are random and have a Gaussian amplitude probability density, the rms value of a Gaussian signal (1σ) would be ($2\sqrt{\pi}$) or approximately 1.3 times the reading of an average sensing voltmeter calibrated in rms for sine waves, Ref. [2]. Then if a Gaussian input signal was adjusted to the maximum indicated input level for such an instrument, clipping would occur for signal amplitudes above only 1.25σ.

Sometimes the problem of limited dynamic range is unavoidable as in the case of the engineer who has no control over sensitivity or gain settings of the instruments during the actual measurements. For missile vibration measurements in particular, unless the test engineer has some prior knowledge of the environment, he often must estimate what the vibration levels will be and pre-set the gain of the measurement instruments accordingly. If the estimate of the environment is poor, the data may be obscured by background noise or distorted by clipping. The problem is further aggravated when the desired vibration response measurements are to cover several different flight phases characterized by different sources of excitation producing widely varying response levels.

9.5 CONCLUSIONS AND RECOMMENDATIONS

The foregoing summary has shown that much effort on analyzing nonlinear systems has been expended during the last few years. It also indicates that some areas appear to be almost completely unexplored. It is realized that a number of investigations exist which contain material not known or discussed herein. Also, many of the references that have been noted contain more material than presented here. However, some general conclusions will be stated as to where considerable work remains to be done.

The area of structural damping particularly appears to need attention. Several companies and universities are presently studying this problem. Exact and approximate analytical solutions are needed for systems with nonlinear damping, different types of stiffness nonlinearities, and combined nonlinear damping and nonlinear stiffness, when these systems are subjected to random forcing functions. Further experimental study of idealized types of structures is also required to determine which analytical models best fit particular types of nonlinear structures when the input is of a random nature.

The following specific recommendations are therefore made for future work.

1. Further experimental study to find proper analytical models for systems with structural damping.

2. Theoretical study of equations of motion with random forcing functions for systems with combined viscous and displacement-cubed damping, and for combined viscous and velocity-squared damping.

3. Theoretical study of equations of motion with random forcing functions for systems having different types of nonlinear stiffness characteristics.

4. Theoretical study of equations of motion with random forcing functions for systems with combined nonlinear stiffness and nonlinear damping.

5. Experimental studies to determine which analytical model best fits various types of nonlinear structures subjected to random inputs

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10. CONCLUSIONS AND RECOMMENDATIONS FROM THEORETICAL PROGRAM

The objectives of the contract and a brief summary of main results are stated in Section 1. Part I of this report deals with the theoretical studies on random signal estimation. To terminate Part I of this report, a section-by-section review will be given here of the theoretical material in Sections 2 through 9, followed by recommendations for further theoretical and experimental work. A review of the experimental work carried out for this contract appears in Section 18 of Part II, together with its own conclusions and recommendations.

10.1 REVIEW OF THEORETICAL PROGRAM

Section 2. Methods for Estimating Mean Values of Nonstationary Data

Results are derived in this section for the accuracy to be associated with mean value (average response) computations on a wide class of nonstationary data as a function of underlying signal and noise vibrations, and the sample size used by the mean value computer. Input and output signal-to-noise ratios are defined in terms of these quantities. Confidence bands are determined for both arbitrary probability distributions and for Gaussian probability distributions, which indicate how closely a measured sample mean value approximates the true mean value at any time. A special result shows the increase in sample sizes required for arbitrary distributions as opposed to Gaussian distributions in order to achieve a desired confidence band. Three cases are developed according to whether or not the sample records are (1) independent, (2) dependent, (3) correlated. The last case includes the other two as special cases. Adaptive changes in records as a result of repeated excitations or stimuli, such as produced by structural fatigue effects or physiological learning patterns, may be analyzed by the mathematical model discussed herein. A quantitative technique is described using an exponential correlation function which can characterize the extent of the correlation between samples of nonstationary data, and thus compare different physical situations.

Section 3. Estimation of Nonstationary Mean Square Values

This section considers the problem of measuring mean square values of nonstationary data. Mean square values represent an important

physical characteristic for describing the "power" contained in the data. Also, mean square values are useful to analyze data where the mean values are zero, a fairly common occurrence with random phenomena. Results are developed along lines similar to the previous section. Three cases are discussed according to whether or not the sample records are (1) independent, (2) dependent, (3) correlated. The variance associated with measurements of mean square values of nonstationary data is determined as a function of underlying input signal and noise variations, and the number of records (sample size) available for the calculations. From these formula, for given input conditions, it is possible to state in statistical terms how well one is able to measure the true mean square value at any time as a function of the sample size. The case of correlated samples provides a method for analyzing fatigue effects or other adaptive physical changes.

Section 4. Spectral Decomposition of Nonstationary Processes

After reviewing the known Fourier transform correspondence between power spectral density functions and autocorrelation functions for stationary random processes, this section extends these concepts to nonstationary random processes. It is shown by proper definitions and interpretation that nonstationary power spectral density functions and nonstationary autocorrelation functions are double Fourier transforms of one another, which reduce to the known formulas in the stationary case. A similar mathematical correspondence is shown to exist for nonstationary cross-power spectral density functions and nonstationary cross-correlation functions. Special situations are considered of separable nonstationary correlation functions and of locally stationary correlation functions where the desired formulas assume simpler forms. Locally stationary processes may be used to represent important physical problems, such as turbulence, where the average instantaneous power may vary slowly with respect to the correlation time. Estimation problems in measurement of nonstationary power spectral density functions are discussed for cases of slowly varying nonstationary random processes.

Section 5. Input-Output Relations for Nonstationary Processes

This section derives mathematical formulas to analyze input-output relations when nonstationary random processes pass through time-varying or constant parameter linear systems. Stability and physical realizability conditions are noted for these systems, as well as certain bandwidth properties. General relationships are found between input and output nonstationary autocorrelation and/or cross-correlation functions, and between input and output nonstationary power spectral density and/or cross-power spectral density functions. The results presented here extend simpler well-known results which apply to stationary random processes. In order to place later material devoted to nonlinear systems on a proper foundation (Section 9), a discussion is given of (1) general linear transformations of random functions, (2) integral transformations, and (3) derivative transformations. In particular, input-output moment transformations are treated. For certain problems, the existence of the derivative of a nonstationary (stationary) random process is shown to depend upon the existence of the first derivative of its nonstationary (stationary) autocorrelation function.

Section 6. Example: Nonstationary Process . . . .

With the aid of formulas derived in the preceding two sections, an example is treated of a nonstationary process resulting from amplitude modulation and filtering of a stationary random process. Besides helping to explain certain physical phenomena which may be generated in this way, this example provides some analytical results which may be applied towards evaluating spectrum simulators. Calculations are carried out of the various stationary and nonstationary auto and cross-correlation functions, and power and cross-power spectral density functions, which occur within the time-varying linear system of this example.

Section 7. Sampling Considerations for Flight Vehicle Vibration Problems

This section begins by reviewing certain material in the previous contract devoted to choosing an appropriate random sampling scheme for decreasing the amount of data to be collected for later analysis.

Considerations involved in periodic sampling of random data are then discussed. For either random sampling or periodic sampling, it is noted when the data must be both random and stationary if the prediction formulas are to be useful. The uncertainty in measured sample values is another factor that should be considered. Thus, one sees the importance of certain experimental results in Part II which have accurately determined expected measurement uncertainties, and which have developed practical statistical tests for randomness and stationarity. Some of these experimental results are summarized here. Examples are given showing how to choose appropriate sampling procedures for different applications. In order to prepare the reader for material in the next Section 8, this section closes by reviewing certain statistical procedures in the previous contract, as well as some other statistical techniques, for analyzing a set of sample values from a single experiment (e. g. , data from a single point). Two cases are developed according to whether or not one includes in the analysis the measurement uncertainties in the sample values.

Section 8. Analysis of Variance Procedures for Evaluating Vibration Data from Many Points

This section presents a direct extension of the basic analysis of variance techniques discussed in the previous contract under the heading of "Repeated Experiments." These procedures supply a classical statistical method of attack for investigating vibration data collected at several points of a structure simultaneously, during one or many flights. Three basic cases of analysis of variance are described. They are: (a) the fixed effects model, (b) the random effects model, and (c) the mixed effects model. In the one-factor or one-way variance analysis, only the first two cases have meaning since only one variable is considered. However, for the two-factor (two-way) variance analysis, two variables are considered which gives rise to the third case. The two-way variance analysis is further broken down into two special cases corresponding to cases where only one observation is available per combination, or where several observations are available per combination. Several comprehensive computational examples are

given illustrating all the fundamental variations. Emphasis is placed upon the proper decision procedures and proper interpretation of results. It should be noted that, for even relatively small amounts of data, the computations involved can become quite extensive. A digital computer is thus seen to be an invaluable tool for carrying out these computations.

Section 9. Response of Nonlinear Systems to Random Excitation

The material presented in this section summarizes and reviews much of the analytical and experimental results available to date on this subject. The response of the "Hardening Spring" to random excitation is covered in considerable detail. This is followed by a discussion of other types of nonlinearities such as the response of nonlinear structures, fatigue of nonlinear structures, the set-up spring, the least-give spring, nonlinear damping, and instrumentation nonlinearities.

10.2 RECOMMENDATIONS FOR FURTHER THEORETICAL AND EXPERIMENTAL WORK FROM PART I

For nonstationary data analysis and applications to vibration problems, as well as to other physical problems, there are three main recommendations for further work:

1. Development of special statistical procedures and design of appropriate experimental programs for analyzing nonstationary data according to formulas derived in Sections 2 through 5 of this report. This should be followed by an experimental program for experimental verification of the validity and practical nature of these techniques, which will yield final improved theoretical methods for measuring desired properties of nonstationary data.
2. Further investigations of different examples of physical systems leading to nonstationary processes as per example in Section 6 of report. Unifying principles should be sought to describe nonstationary response behavior and transfer characteristics for special categories of systems, both linear and nonlinear. This should lead to better predictions of expected nonstationary environments, with more accurate design and simulation requirements for systems which must operate in such environments.

3. Studies on accuracy in estimating nonstationary cross-correlation functions for different mathematical models, and research on suitable physical applications of these methods. These applications should yield improved detection methods for separating signals from noise, and should yield improved techniques for predicting desired properties of the data being analyzed.

To advance other work in Sections 7, 8, and 9 of this report, concerned with estimating various flight vehicle vibration properties, and nonlinear systems, the following recommendations are made:

1. Design and carry out an appropriate experimental program to verify the sampling methods of Section 7, and analysis of variance techniques of Section 8, as applied to vibration data. This would mainly be for establishing the suitability of random or periodic sampling, and the appropriateness of underlying mathematical variance models.
2. Design and develop necessary digital computer programs for the data reduction associated with the sampling methods and with the analysis of variance procedures. The input and output features of the program should take into account special characteristics of vibration data.
3. Perform further theoretical work aimed at simplification of the sampling methods and the analysis of variance procedures. This simplification should be guided and influenced by the experimental investigations.
4. Develop and experimentally verify appropriate mathematical models,as suggested in Section 9, for analyzing continuous structures with nonlinear stiffness and/or nonlinear damping characteristics.

PART II

EXPERIMENTAL STUDIES OF RANDOM SIGNAL MEASUREMENTS

Special Notice

The experiments discussed herein involve many different types of commercial instruments. In order to accurately and completely describe the details of the experiments performed, the instruments employed for each experiment are identified by manufacturer and model number. It is to be clearly understood that the actual instruments used for each experiment were selected solely on a basis of availability. These instruments should not be regarded as either superior or inferior to any other commercial instruments which will accomplish the same desired functions.

11. INTRODUCTION TO EXPERIMENTAL PROGRAM

11.1 GENERAL BACKGROUND

When flight vehicle vibration data is gathered and analyzed, the usual objective is to obtain information concerning the vibration environment to be expected during future missions for that and all similar vehicles. Given a sample vibration response amplitude* time history record obtained during the flight of a given vehicle, one may readily measure various descriptive properties of the recorded vibration response, such as amplitude probability density functions, correlation functions, power spectral density functions, etc. However, these measurements describe only the vibration response in that vehicle for that interval of time in the past when the sample record was obtained. If the measurements are to be of value for predicting the vibration environment during future missions of that and other similar vehicles, certain fundamental characteristics of the vibration environment must be considered. Three such characteristics are randomness, stationarity, and normality.

If the vibration environment is random and stationary, the descriptive properties measured from recorded samples of the vibration response will constitute estimators for the true properties of the response. The sampling errors for these measurements, that is the uncertainties of the estimates, are vitally important to the accuracy of conclusions which will be drawn from the data.

These matters have been considered theoretically in the document ASD TR 61-123, "The Application of Statistics to the Flight Vehicle Vibration Problem." Quantitative procedures for testing sample vibration response records for randomness, stationarity, and normality are suggested in Section 6 of that document. The techniques for estimating the descriptive properties of a vibration response, such as power spectra

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*Vibration response amplitudes are usually obtained in the form of voltage signals from transducers, and may be a measure of any physical response parameter (acceleration, velocity, displacement, stress, etc.) depending upon the nature of the transducer employed. The particular response parameter being measured is of no direct importance to the analysis procedures discussed herein.

and probability density functions, and the uncertainties associated with their estimation are discussed in Section 7 of that document. Many of these theoretical ideas are now to be studied experimentally.

11.2 UNCERTAINTIES OF VIBRATION RESPONSE MEASUREMENTS

When the descriptive properties of a vibration response at some point on the structure of a flight vehicle are determined from a sample record, an important question arises. How accurately do the properties determined from the sample record represent the actual properties of the vibration response being sampled? The expected deviation of determined properties from actual properties of the vibration response may be generally thought of as the data analysis error.

When a vibration response is periodic in nature, a sample record with a length of only one period is theoretically sufficient to define the exact vibration amplitude that would exist at any time assuming identical conditions. Thus, the only limitations upon the accuracy of periodic vibration response measurements are the errors associated with the instruments and calibration procedures employed, which will be called measurement errors. Measurement errors consist of all factors in the translation of a vibration response into final data which produce deviations from an exact or mathematically precise translation. Included are amplitude nonlinearities, imperfect frequency response, sensitivity drift, etc., in the data acquisition and analysis instruments, as well as observation errors in any visual readouts. The manufacturers of vibration transducers, transmission systems, recorders, and analyzers usually publish the accuracy limits that their instruments are designed to meet.

For a random vibration response (assumed to be stationary), the amplitude at any instant of time in the past or future supplies information concerning the properties of the vibration. Even if the measurement procedure employed were mathematically precise (no measurement errors), the exact properties of the vibration response could be established only by investigating the amplitudes over all time. The data obtained from sample records of finite length constitute only statistical estimates of the properties of the vibration. The expected

deviation of the estimated properties from the actual properties of a random vibration response may be thought of as an error, which will be called the sampling error or uncertainty. The accuracy of the estimation, as well as the accuracy of the measurement, must always be considered in the analysis of a random vibration response from sample records.

Measurement errors are determined by the specific instruments and calibration procedures used. On the other hand, sampling errors or estimation uncertainties are a function of the length of sample records selected for analysis and inherent properties of the vibration response being analyzed. It is the sampling errors or uncertainties that are of concern in these experimental studies.

The simplest property used to describe a single random vibration response is the mean square or root mean square value. However, this yields only a rudimentary definition of the vibration amplitude. A more detailed description is given by two important properties, the power spectral density function and the amplitude probability density function. A third important property is the autocorrelation function which, for a stationary vibration response, is the inverse Fourier Transform of the power spectrum. A fourth property of interest is the expected number of zero crossings per second.

The power spectrum of a stationary random vibration response is a measure of the relative power per cps versus frequency and has the units $volts^2/cps$; i.e., g^2/cps, $inches^2/cps$, psi^2/cps, etc. Analytically, the power spectrum of a stationary random signal is the Fourier Transform of the autocorrelation function of the signal. Hence, the power spectrum can theoretically be obtained either directly by filtering in the frequency domain or indirectly by filtering in the time domain (autocorrelation) and determining the Fourier Transform of the results. These studies are concerned with the estimation of power spectra by direct frequency filtering. Autocorrelation analysis is considered only in qualitative terms.

The probability density function of a stationary random vibration response defines the likelihood of given amplitude occurrences. It gives a complete description of the vibration amplitude as opposed to

the mean square value which yields only a rudimentary measure of vibration amplitude. The expected number of zero crossings per second for a stationary vibration response presents interesting implications concerning the random nature of the vibration.

Sections 12, 13, and 14 herein are concerned with the uncertainties of zero crossing, power spectra, and probability density estimates resulting from sample record measurements. Section 12 deals with zero crossing estimates, Section 13 with power spectra estimates, and Section 14 with probability density estimates. Estimation uncertainties are discussed before the tests for fundamental characteristics because uncertainty considerations form a basis for these tests.

11.3 FUNDAMENTAL CHARACTERISTICS OF VIBRATION ENVIRONMENTS

Given the task of analyzing a flight vehicle vibration environment, the first question which should be answered is as follows. Is the vibration response being measured random in nature as opposed to periodic? Perhaps both random and periodic components are present. The desired procedures for analyzing and describing the vibration environment may be quite different for each case. The techniques for applying the end data to design problems are also affected. Periodic vibrations can be described by explicit analytic functions while random vibrations must be described by statistical functions. As a result, the length of sample records to be gathered for analysis is more critical for random vibrations due to inherent statistical uncertainties or sampling errors in the resulting measurements.

If a vibration environment is random in nature, the next fundamental characteristic of interest is stationarity. Is there reason to believe that the vibration response being measured will exist in the future under conditions identical to those which existed when a sample record was obtained for analysis? Clearly, the problem of predicting future events is directly dependent upon the stationary characteristics of the vibration environments being measured and analyzed.

The third fundamental characteristic to be considered is normality. For a random vibration, do the instantaneous vibration amplitudes have a Gaussian probability density function? If the

amplitude distribution for a random vibration environment is found to be sufficiently Gaussian to justify the normality assumption, the application of the resulting data to design problems is greatly simplified. Furthermore, the existence of normality will greatly increase the power of conclusions concerning the stationary characteristics of the vibration environment.

During the measurement and analysis of a flight vehicle vibration environment, the characteristics of randomness, stationarity, and/or normality are often assumed without quantitative justification. In many cases, strong qualitative arguments can be developed which tend to justify any or all of the assumptions. For example, if one can substantiate that the exciting forces to be expected during a mission are basically stochastic processes (such as aerodynamic boundary layer turbulence and jet exhaust gas mixing), it appears appropriate to assume that the vibration responses will be primarily random in nature. If it is known that at least certain phases of a mission will occur under reasonably constant conditions (airspeed, altitude, thrust, etc.), there appears to be no reason to question that the vibration responses will be stationary during such phases. If the vibration environment is random, the practical implications of the Central Limit Theorem in statistics tend to support an assumption that the instantaneous vibration amplitudes will be distributed in a Gaussian manner.

On the other hand, unknown or overlooked factors may destroy the validity of these assumptions. For example, a flight vehicle may carry equipments which generate intense periodic vibrations in local areas. Unexpected events may occur during a mission which produce extreme vibration levels for short intervals during a period of assumed stationary conditions. Lack of normality in the exciting forces and/or nonlinear characteristics of the structure may result in vibration response amplitude distributions which deviate significantly from the ideal Gaussian form.

It is important that fundamental assumptions such as randomness, stationarity, and normality, be confirmed when possible by investigation of the sample vibration response records gathered for analysis.

Sections 15, 16, and 17 herein are concerned with the determination of fundamental characteristics of a vibration response by tests applied to sample record measurements. Section 15 deals with tests for randomness, Section 16 with tests for stationarity, and Section 17 with tests for normality.

11.4 DEFINITIONS OF FUNDAMENTAL CHARACTERISTICS

It is appropriate here to define the terms random, stationary, and normal as they are used in these studies. The definitions presented are limited and not necessarily rigorous. More discussions and definitions of these fundamental characteristics are available in ASD TR 61-123.

11.4.1 Random Vibration

Random vibration is that type of time-varying motion which consists of randomly varying amplitudes and frequencies such that its behavior can be described only in statistical terms. No analytical representation for the motion is possible. The motion does not repeat itself in finite time periods. In this report, a vibration response will be considered as random unless vibratory motion with a periodic form is present. There are, of course, other types of vibratory motions that are neither random nor periodic. Included would be those motions which can be represented by nonperiodic analytic functions, such as an exponential-sine function. However, it is usually more suitable to consider and analyze these motions as shock phenomena rather than as vibration.

11.4.2 Stationary Random Vibration

The term stationary is often applied to random vibration data in a context that is slightly different from its classical statistical meaning. Because the concept of stationarity is so vitally important to vibration environment prediction problems, some general discussion of its meaning is included here.

Any particular random vibration response, whose amplitude time history has been recorded for analysis, represents a unique set of circumstances that are not likely to ever be repeated exactly. A single random vibration record, in actual practice, is merely a special example out of a large set of possible records that might have occurred. This collection (ensemble) of records may be thought of as a random process, as illustrated in Figure 11.1.

Hypothetically, the number of records in a random process would be infinitely large and each record would exist over all time. The properties of the random process may be computed by taking averages over the collection of records at any given time t. Such averages, called ensemble averages, will be a function of time. For example, at time t, the mean value $\mu_x(t)$, the mean square value $ms_x(t)$, and the autocorrelation function $R_x(t, \tau)$ would be given by

$$\mu_x(t) = \lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} x_k(t) \qquad (11.1a)$$

$$ms_x(t) = \lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} x_k^2(t) \qquad (11.1b)$$

$$R_x(t, \tau) = \lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} x_k(t) x_k(t+\tau) \qquad (11.1c)$$

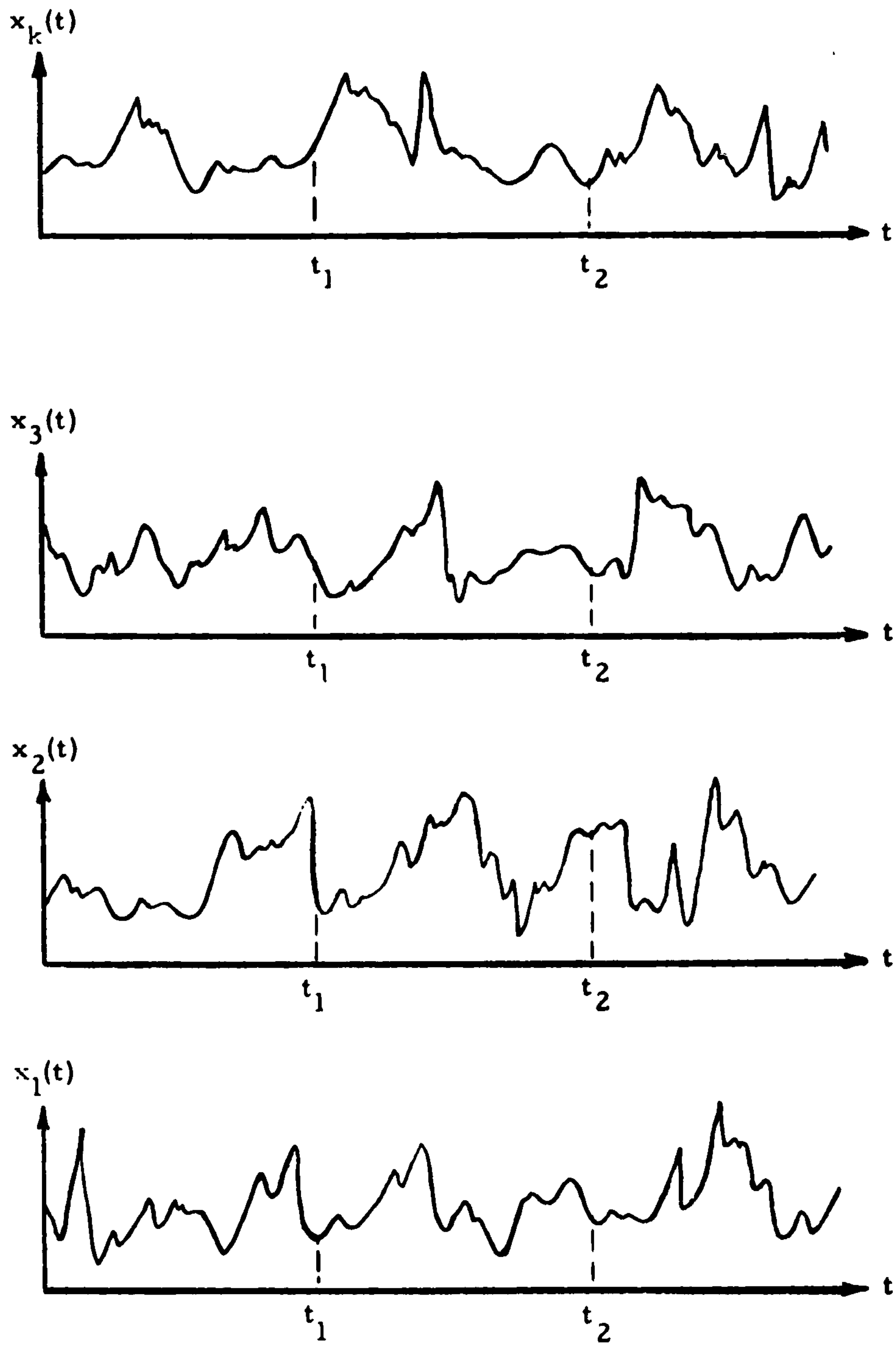


Figure 11.1 Vibration Amplitude Time History Records

For the general case where the properties defined in Eq. (11. 1) vary with time, the random process is said to be nonstationary. For the special case where the properties defined in Eq. (11. 1) do not vary with time (μ_x, ms_x, and $R_x(\tau)$ are time invariant), the random process is said to be <u>weakly stationary</u>. If all possible moments of the random process determined by ensemble averaging are time invariant, the random process is said to be <u>strongly stationary</u>. In its broadest meaning, stationarity is a characteristic of ensemble averaged properties.

The properties of individual records within the random process can also be computed by averaging over time. These time averages will be a function of the specific record in the collection. For the ith record, the mean value $\mu_x(i)$, the mean square value $ms_x(i)$, and the autocorrelation function $R_x(\tau, i)$, would be given by

$$\mu_x(i) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x_i(t)\, dt \tag{11. 2a}$$

$$ms_x(i) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x_i^2(t)\, dt \tag{11. 2b}$$

$$R_x(\tau, i) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x_i(t) x_i(t+\tau)\, dt \tag{11. 2c}$$

Let the above time averaged properties be determined for all possible records of a random process which is at least weakly stationary. If each property is the same for all records and equal to the corresponding property determined by ensemble averaging, the stationary random process is said to be <u>weakly ergodic</u>. If the random process is strongly stationary, and the above requirements are met for all possible moments, the stationary random process is said to be <u>strongly ergodic</u>. Thus, ergodicity is a characteristic of time averaged properties for a stationary random process.

For the flight vehicle vibration problem, a single record could represent the vibration response at some point on the structure of a given flight vehicle. The collection of records (ensemble) would then represent the vibration responses at that point occurring during simultaneous flights of all vehicles of that type. The procedure of measuring descriptive properties of the vibration response by ensemble averaging is clearly undesirable. In order to minimize the statistical uncertainty (sampling error) in the resulting measurements, a relatively large number of simultaneous flights of different vehicles (or repeated flights of the same vehicle) would be required. If data is reduced by analog instruments, the instrumentation required to ensemble average a large number of sample records simultaneously is far more complex than is required to time average the individual sample records. It is for these reasons and others that the vast majority of vibration data analysis, in actual practice, is accomplished by time averaging individual sample records rather than by ensemble averaging a collection of sample records. A slightly different concept for stationarity is required when one deals with vibration response data by time averaging individual sample records.

An individual vibration response is often referred to as being stationary or nonstationary. In this case, the term is used to mean that the properties of the vibration response, determined by time averaging sample records of finite length, do not change "significantly" for samples covering different time intervals. The word significantly means that variations in the sample values are greater than would be expected from sampling errors. This concept of stationarity will, for the moment, be referred to as self-stationarity to avoid confusion with the definition presented previously.

To clarify the idea of self-stationarity, consider an individual vibration response $x(t)$. Assume a sample record of length T is obtained at some starting time t_0. In general, the properties determined by time averaging over the sample will be a function of

the starting time t_0. That is, for time t_0, the sample mean value $\bar{x}(t_0)$, the sample mean square value $\overline{x^2}(t_0)$, and the sample autocorrelation function $\hat{R}(\tau, t_0)$ would be given by

$$\bar{x}(t_0) = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\,dt \tag{11.3a}$$

$$\overline{x^2}(t_0) = \frac{1}{T}\int_{t_0}^{t_0+T} x^2(t)\,dt \tag{11.3b}$$

$$\hat{R}(\tau, t_0) = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)x(t+\tau)\,dt \tag{11.3c}$$

For the general case where the sample properties defined in Eq. (11.3) vary significantly with the time t_0 that the sample was obtained, the individual vibration response is said to be self-nonstationary. For the special case where the properties defined in Eq. (11.3) do not vary significantly with time t_0, the vibration response is said to be <u>weakly self-stationary</u>. If the above requirement is met for all possible moments, the vibration response is said to be <u>strongly self-stationary</u>.

It should be noted that if a vibration response is normally distributed, the first two moments (mean value and mean square value) will define all higher moments. Thus, if it can be shown that the vibration response is weakly self-stationary, this fact plus the existence of normality makes the response strongly self-stationary.

There are many possible associations between the self-stationarity of individual vibration responses, and the stationarity and ergodicity of a collection of vibration responses. The most important of these associations is as follows. Assume the vibration responses in many different vehicles of the same type are each found to be self-stationary, at least during some phase of a common mission. Also assume the properties of the vibration responses determined by time averaging sample records obtained during a self-stationary phase are equivalent from vehicle to

vehicle. Then, the vibration response during that phase is stationary and ergodic. The properties determined by time averaging a short sample record from one vehicle will be estimators for the vibration response to be expected in all vehicles of that type during the entire phase.

11.4.3 Normal (Gaussian) Random Vibration

A random vibration response x(t) is Gaussian if it has a specific instantaneous amplitude probability density function given by

$$p_o(x) = \frac{1}{\sigma_x \sqrt{2\pi}} e^{-(x-\mu_x)/2\sigma_x^2} \qquad (11.4)$$

Here, μ_x is the mean value for the response, and σ_x^2 is the mean square value about the mean (variance) for the response. Random vibration responses which are described by any other probability density function will be called non-Gaussian.

11.5 MEAN SQUARE VIBRATION RESPONSE

For any random signal, the mean square value, msv_x, and the mean square value about the mean (variance), σ_x^2, are related by $\sigma_x^2 = msv_x - \mu_x^2$. Thus, if the mean value μ_x is zero, the two quantities are equal, and σ_x^2 defines the mean square value of the signal. The positive square root, σ_x, is the root mean square value of the signal.

It will be assumed in all discussions to follow that the mean value, μ_x, for the vibration response is zero. The mean value of a vibration response is a measure of the steady state magnitude for the physical parameter being measured (such as steady state acceleration), and would appear as a DC voltage in the output of the vibration transducer. Because of the limited low frequency response of most vibration data acquisition and analysis equipment, the mean value of the vibration response is usually rendered zero whether the mean value of the actual physical parameter being measured is zero or not. Hence, the term σ_x^2 will be used in these studies to denote the mean square value of a vibration response, as is done in ASD TR 61-123, where it is understood that the mean value of the vibration response is assumed to be zero.

12. UNCERTAINTY OF ZERO CROSSING ESTIMATES

12.1 THEORY OF ZERO CROSSING ESTIMATION

12.1.1 General Remarks

A procedure for testing the randomness of vibration amplitude time history records is suggested in Section 6.1.5 of Ref. [1]. The proposed test for randomness involves the application of statistical run theory to the prediction of zero crossings for continuous vibration signal records. Before that test can be considered in more detail, it is necessary to confirm the validity of theoretical predictions for zero crossings of random signals. The actual proposed test for randomness using zero crossing data is studied in Section 15 of the report.

12.1.2 Review of Basic Concepts

Consider an event which can occur in only two mutually exclusive ways; that is, the outcome must be either A or B. For example, the event may be the flip of a coin where the outcome will be either a head (A), or a tail (B). As another example, the event may be a gage pressure reading where the outcome will be either a positive pressure (A) or a negative pressure (B) relative to gage reference. Now assume the experiment producing the event is repeated n number of times in sequence, such as a coin being flipped 20 times in succession. The outcomes of the sequence of events might be as follows.

$$\underbrace{AA}_{1}\quad \underset{2}{B}\quad \underset{3}{A}\quad \underbrace{BB}_{4}\quad \underbrace{AAA}_{5}\quad \underset{6}{B}\quad \underset{7}{A}\quad \underbrace{BB}_{8}\quad \underset{9}{A}\quad \underbrace{BB}_{10}\quad \underset{11}{A}\quad \underbrace{BBB}_{12}$$

A run is defined as a sequence of identical outcomes which are followed or preceded by a different outcome or no outcome at all. In the above example the sequence of 20 events resulted in 12 runs. The total number of runs in any observed sequence of events gives an indication of whether or not the events occur randomly.

Given a sample of n number of events, let n_1 be the number of events with an outcome A, and let n_2 be the number of events with an outcome B. Clearly, $n = n_1 + n_2$. Also, let r be the number of runs which occur in the sample. If the events occur randomly, and if n_1 and n_2 are greater than 20, it is shown in Ref. [3] that a good approximation for the

sampling distribution of r is a normal distribution with a mean of μ_r and a variance of σ_r^2 given by,

$$\mu_r \approx \left(\frac{2n_1 n_2}{n} \right) + 1 \tag{12.1}$$

$$\sigma_r^2 \approx \frac{2n_1 n_2 (2n_1 n_2 - n)}{n^2 (n - 1)} \tag{12.2}$$

12.1.3 Development for Vibration Signal Records

Consider a sample vibration amplitude time history record of length T, as shown in Figure 12.1.

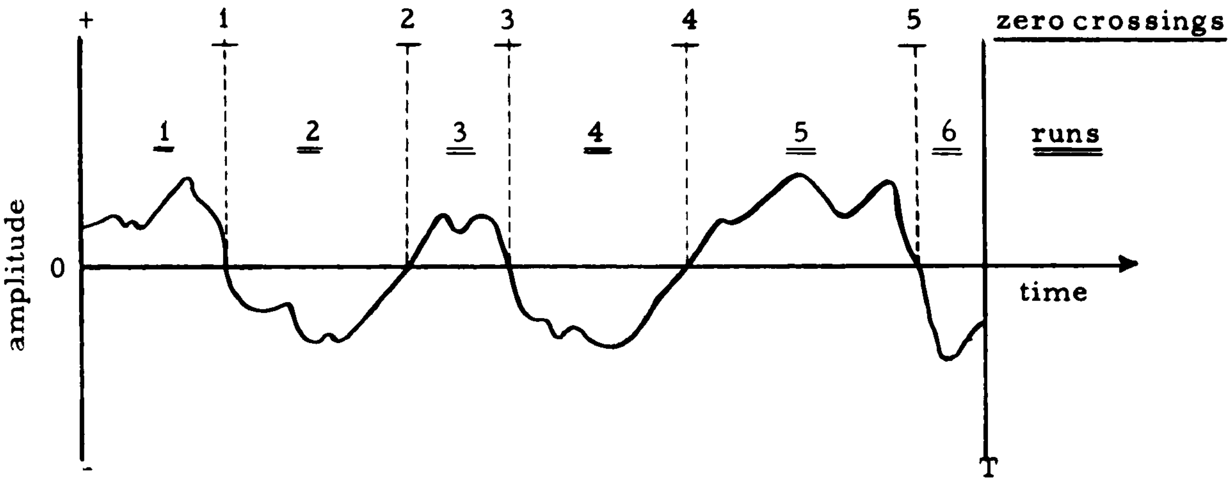


Figure 12.1 Sample Vibration Amplitude Time History Record

For simplicity, the mean value of the vibration signal being sampled is assumed to be zero; i.e., no DC component is present. This is usually the actual case in practice.

Assume that the total number of events represented by the sample record of length T is some number n, still to be defined. Let the two possible outcomes of each event be either a positive amplitude value or a negative amplitude value. Then, a zero crossing constitutes the end of one run and the beginning of another, as shown in Figure 12.1. The total number of runs r in a given sample record will always be one greater than the total number of zero crossings ν_o. That is,

$$r = \nu_o + 1 \tag{12.3}$$

Furthermore, the total time that the signal has a positive amplitude value will be approximately equal to the total time that the signal has a negative amplitude value. Then, it may be assumed that

$$n_1 \approx n_2 \approx \frac{1}{2} n \tag{12.4}$$

When applying these ideas to actual vibration signals, one would determine the number of runs in a sample record by counting the number of zero crossings that occur over the record length. Thus, it will be more practical to work directly with zero crossings rather than runs. Noting Eq. (12.3), the mean and variance of zero crossings are related to the mean and variance of the runs as follows.

$$\mu_{\nu_o} = \mu_{(r-1)} = \mu_r - 1 \tag{12.5}$$

$$\sigma^2_{\nu_o} = \sigma^2_{(r-1)} = \sigma^2_r \tag{12.6}$$

If the vibration signal being sampled is random, by substituting Eqs. (12.1), (12.2), and (12.4) into Eqs. (12.5) and (12.6), the sampling distribution for zero crossings will have a mean and variance given by

$$\mu_{\nu_o} \approx \frac{n}{2} \tag{12.7}$$

$$\sigma^2_{\nu_o} \approx \frac{n\left(\frac{1}{2} n - 1\right)}{2(n-1)} \tag{12.8}$$

It now remains to define the number of events n that are represented in a sample record of length T.

In Ref. [1], the number of events n was considered to be the number of degrees of freedom of the sample record given by $n = 2BT$, where B is the frequency bandwidth in cps after filtering by an ideal low pass filter with an infinitely sharp cutoff. However, this definition for n is not consistent with the results of theoretical studies of zero crossings for random signals. From Ref. [2], the expected number of zero crossings for a random signal record of length T seconds with a low pass frequency bandwidth of B cps is given by

$$\mu_{\nu_o} \approx 1.15\, BT \tag{12.9}$$

If the number of events is defined as $n = 2BT$, from Eq. (12. 7) the expected number of zero crossings would be

$$\mu_{\nu_o} \approx 1.00\ BT \tag{12.10}$$

Because the problem at hand is directly concerned with zero crossings, the results presented in Ref. [2], as stated in Eq. (12. 9), will be accepted as a basis for defining the number of events n. The validity of this decision will be substantiated by experimental results to be presented later.

Again referring to Ref. [2], the general equation for the expected number of zero crossings per second, $\overline{\nu}_o$, for a random signal with a frequency bandwidth between the limits f_a cps and f_b cps is given by

$$\overline{\nu}_o = 2\sqrt{\frac{f_a^2 + f_a f_b + f_b^2}{3}} \tag{12.11}$$

where it is assumed that the random signal has the following properties.

(a) an infinitely sharp cutoff at frequencies f_a and f_b.

(b) a uniform power spectral density function over the frequency range between f_a and f_b.

(c) a Gaussian amplitude probability density function.

If the random signal being investigated does not possess the above characteristics, the equation for expected zero crossings per unit time becomes somewhat more complex. However, Eq. (12. 11) may still be an acceptable approximation for practical purposes. The importance of the above assumptions will be considered in more detail later.

From Eq. (12. 11), the expected number of zero crossings in a sample record of length T is

$$\mu_{\nu_o} = \overline{\nu}_o T \tag{12.12}$$

Comparing Eq. (12. 12) with Eq. (12. 7), the number of events n in a sample record of length T may then be defined as

$$n = 2\overline{\nu}_o T \tag{12.13}$$

where $\overline{\nu}_o$ is given by Eq. (12. 11).

12.2 DESIGN OF EXPERIMENTS AND PROCEDURES

12.2.1 General Design and Procedures

The experiments to be conducted are designed to investigate the sampling distribution of zero crossings and confirm the validity of the theoretical results given in Eqs. (12.7) and (12.8). Experiments are also designed to study the practicability of the theoretical results in terms of simulated vibration data. The sampling distribution of zero crossings are investigated for random and non-random signals in three basic areas.

Area A. random signals which have approximately Gaussian probability density functions and uniform power spectral density functions.

Area B. random signals which have approximately Gaussian probability density functions and non-uniform power spectral density functions.

Area C. non-random signals composed of a sine wave mixed with a random signal.

For all three areas of study, the basic procedure is to gather some large number N of sample records from a simulated vibration signal of interest. The number of zero crossings, ν_o, in each of the N number of samples is counted. The mean value and variance of the sampling distribution are then estimated as follows:

$$\hat{\mu}_{\nu_o} = \frac{1}{N}\sum_{i=1}^{N} \nu_{o_i} \tag{12.14}$$

$$\begin{aligned} \hat{\sigma}^2_{\nu_o} &= \frac{1}{N}\sum_{i=1}^{N} (\nu_{o_i} - \hat{\mu}_{\nu_o})^2 \\ &= \frac{1}{N}\left[\sum_{i=1}^{N} \nu_{o_i}^2 - \frac{\left(\sum_{i=1}^{N} \nu_{o_i}\right)^2}{N}\right] \; * \end{aligned} \tag{12.15}$$

---

* A biased variance estimate is employed here so that all statistical procedures to follow will be consistent with procedures outlined in Ref. [1].

For experiments in Area A, if the theoretical expected value of the sampling mean μ_{ν_o} defined by Eq. (12. 7) is valid, the estimated sampling mean $\hat{\mu}_{\nu_o}$ given by Eq. (12. 14) should be statistically equivalent to μ_{ν_o}. Likewise, the estimated sampling variance $\hat{\sigma}^2_{\nu_o}$ from Eq. (12. 15) should be statistically equivalent to the theoretical expected value for the sampling variance $\sigma^2_{\nu_o}$ defined by Eq. (12. 8). The equivalence of the experimentally estimated and theoretical expected values are studied by testing the null hypotheses, $H_o(\sigma^2)$ and $H_o(\mu)$, where

$$H_o(\sigma^2) \quad : \quad \hat{\sigma}^2_{\nu_o} = \sigma^2_{\nu_o}$$

$$H_o(\mu) \quad : \quad \hat{\mu}_{\nu_o} = \mu_{\nu_o}$$

The null hypothesis, $H_o(\sigma^2)$, is tested by a chi-square test of variances. If $H_o(\sigma^2)$ is accepted, there is no reason to question that $\hat{\sigma}^2_{\nu_o} = \sigma^2_{\nu_o}$, and the theoretical value for $\sigma^2_{\nu_o}$ defined in Eq. (12. 8) is considered valid. If $H_o(\sigma^2)$ is rejected, there is reason to question that $\hat{\sigma}^2_{\nu_o} = \sigma^2_{\nu_o}$, and the theoretical value for $\sigma^2_{\nu_o}$ is questioned. Assuming $H_o(\sigma^2)$ is accepted, $H_o(\mu)$ is then tested by a Normal test of means. The theoretical expected value for μ_{ν_o} defined in Eq. (12. 7) is considered valid if $H_o(\mu)$ is accepted, or questioned if $H_o(\mu)$ is rejected. Note that the variances must be tested first because the Normal test of mean values requires a known variance. However, if $H_o(\sigma^2)$ is rejected, $H_o(\mu)$ can still be tested by a student's "t" test which uses the sample variance.

Assuming the validity of Eqs. (12. 7) and (12. 8) are confirmed by experiments in Area A, the procedure for testing the null hypotheses is continued for the results of experiments in Areas B and C. The practicability of the theoretical ideas will be indicated in part by whether or not the

null hypotheses are accepted or rejected when applied to results in Area B. Clearly, it is hoped that the null hypotheses will be rejected for results in Area C.

The Type I and Type II errors associated with the null hypothesis tests are, of course, a function of the number N of samples gathered and tested. For all experiments the number of samples gathered is chosen to be N = 31. The hypothesis tests are applied at the 10% level of significance, giving a probability of a Type I error of $\alpha = 0.10$. The associated probability of a Type II error is $\beta = 0.10$ for detecting a 6% difference in means and 2.4 to 1 difference in variances (50% difference in standard deviations). The development of these probable errors and the sample size of N = 31 is presented in Section 12.2.2.

12.2.2 Sample Size Calculations for Equivalence of Means and Variances

In any situation where one knows or hypothesizes a mean and variance of a distribution, one may compute sample sizes necessary to properly test sample values against these theoretical values. Certain constraints must be imposed on the problem, such as specifying the level of significance and probability of Type II error, which will be explained below. The sample size may then be calculated which maintains these probabilities.

A theoretical mean μ_{ν_0} and variance $\sigma_{\nu_0}^2$ can be computed for a distribution of zero crossings from Eqs. (12.7) and (12.8). Using these theoretical values, one can then test obtained sample values against these to determine if the observed distribution of zeros can be considered to be the same as the theoretical distribution. That is, the statistical hypothesis will be: there is no evidence to conclude that the sample values are not the same as the theoretical values. These will be two-tailed tests since deviations may occur in either direction

Two types of errors can be made:

Type I error - rejecting the hypothesis when it is really true with probability α

Type II error - accepting the hypothesis when it is really false with probability β

In addition to specifying α and β, one must impose an additional restraint on the problem to allow the sample size to be calculated. That is,

one must specify what particular deviation from the hypothesized parameter will allow the hypothesis to be accepted with probability β. In some specific situations one might have suspicions about the theory involved and anticipate some particular value other than the hypothesized value. In other cases, one must use judgment in selecting values somewhat arbitrarily.

For the experimental program, a ten percent difference in means and a fifty percent difference in standard deviations will be selected as the values at which the probability of Type II error will be held. Of course, other deviations from the theoretical values may be chosen, and have a specific associated probability of the hypothesis being accepted. Also, for simplicity, α and b will be chosen each equal to 10%.

(a) Sample Size for Equivalence of Means

The calculation of the required sample size for the test of equivalent mean values is as follows: Let μ' be the mean value of the distribution which is to be detected with a probability $\beta = 10\%$, and $Z_{1-\alpha/2}$ a normal (Gaussian) deviate such that

$$\text{Prob}(Z \geq Z_{1-\alpha/2}) = \alpha/2$$

The reason for using $\alpha/2$ instead of α is to allow for two-sided deviations so that the Type I error here is α.

The following relations now hold where x_c is the critical point (see Figure 1). That is, x_c is a value such that if $\bar{x} \geq x_c$, the hypothesis is rejected, and if $\bar{x} < x_c$, the hypothesis is accepted where $\bar{x}$ is the calculated sample mean.

Figure 1 is not quite complete in the sense that deviations in either direction are considered. However, the symmetry is implied by using $\alpha/2$ and $\beta/2$ rather than α and β. In terms of $\alpha/2$,

$$Z_{1-\alpha/2} = \frac{x_c - \mu}{\sigma/\sqrt{N}} \tag{12.16}$$

while, in terms of $\beta/2$,

$$Z_{\beta/2} = \frac{x_c - \mu'}{\sigma/\sqrt{N}} \tag{12.17}$$

where σ^2 is the theoretical variance, Eq. (12.8), of the distribution of zero crossings (the subscripts ν_o are being omitted for simplicity).

$$\text{Prob}\,(x \geq x_c) = \alpha/2 \quad ; \quad \text{Type I error}$$

$$\text{Prob}\,(x \leq x_c) = \beta/2 \quad ; \quad \text{Type II error}$$

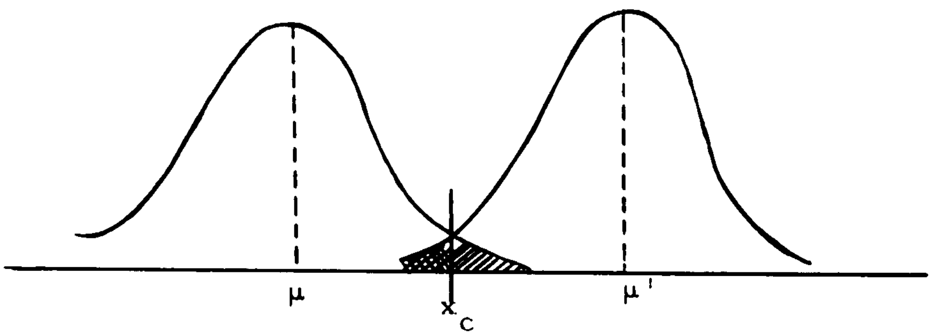


Figure 12. 2 Illustration of Type I, Type II Errors

In the special situation where one sets $\alpha = \beta$, it follows that $Z_{\alpha/2} = Z_{\beta/2}$. Also, due to the symmetry of the normal distribution, $Z_{1-\alpha/2} = -Z_{\alpha/2}$. Hence, from Eqs. (12.16) and (12.17),

$$\frac{x_c - \mu}{\sigma/\sqrt{N}} = -\frac{x_c - \mu'}{\sigma/\sqrt{N}}$$

Solving for x_c one obtains

$$x_c = \frac{\mu + \mu'}{2} \tag{12.18}$$

Then substituting back in Eq. (12.17), replacing β by α,

$$Z_{\alpha/2} = \frac{\frac{\mu + \mu'}{2} - \mu'}{\sigma/\sqrt{N}} = \frac{\frac{\mu - \mu'}{2}}{\sigma/\sqrt{N}} = -Z_{1-\alpha/2}$$

Letting $\Delta\mu = \mu - \mu'$ and solving for N:

$$N = \frac{\sigma^2 Z^2_{\alpha/2}}{(\Delta\mu/2)^2} = 4\left(\frac{\sigma}{\Delta\mu} \; Z_{\alpha/2}\right)^2 \tag{12.19}$$

Eq. (12.19) is a general result which is applicable to many problems. Note that the implicit assumption has been made that $\sigma^2 = (\sigma')^2$ where $(\sigma')^2$ is the alternative variance. Therefore this test would be performed after the alternative variance $(\sigma')^2$ had been determined to be statistically equivalent to the theoretical value σ^2.

(b) Computational Example

The calculation of N based on the test for equivalent means is as follows. For n = 100 effective events, one obtains from Eqs. (12.7) and (12.8) the theoretical values

$$\mu = 50$$
$$\sigma^2 = 25$$

The required sample size to detect a 10% difference in means (namely $\Delta\mu = 5$ here) with a Type II error of $\beta = 10\%$ then is calculated by applying Eq. (12.19). For $\alpha = 10\%$, the term $Z_{.05} = -1.645$. Thus, from Eq. (12.19),

$$N = 4\left(\frac{25}{25}\right)(1.645)^2 \approx 11$$

This sample size of N = 11 will be compared later to the sample size required for equivalent variances.

(c) Sample Size for Equivalence of Variances

The reasoning for obtaining a formula to compute the sample size for the variance equality test proceeds in a similar manner. Let s_c^2 represent the critical point. Then, since $N s^2/\sigma^2$ has a χ^2 distribution with (n - 1) d.f., one has

$$s_c^2 = \frac{\sigma^2}{N} \chi^2_{\alpha/2} \tag{12.20}$$

and

$$s_c^2 = \frac{(\sigma')^2}{N} \chi^2_{1-\beta/2} \tag{12.21}$$

where $\chi^2_{\beta/2}$ and $\chi^2_{1-\alpha/2}$ are points of the χ^2 distribution with (N - 1) d. f. Equating (12. 20) and (12. 21) and rearranging terms, one obtains the general result

$$\frac{(\sigma')^2}{\sigma^2} = \frac{\chi^2_\alpha}{\chi^2_{1-\alpha/2}} \tag{12.22}$$

for the case when $\alpha = \beta$. Although the N cancels out, χ^2 is a function of (N - 1). Therefore, when σ^2 and $(\sigma')^2$ are specified, a trial and error inspection of a χ^2 table will give values of χ^2 for some number of d. f. such that Eq. (12. 22) holds true.

(d) Computational Example

For example, for (N - 1) = 30 d. f. and α = 10%, one finds in the χ^2 table

$$\frac{\chi^2_{.05}}{\chi^2_{.95}} = \frac{43.8}{18.5} = 2.37$$

For all practical purposes this corresponds to the desired ratio of standard deviations of 1. 5 to 1. 0. Therefore, a convenient sample size for the experiments is 31.

Note that the variance equivalence test has a larger required sample size than the mean equivalence test, namely N = 31 as compared to N = 11, and therefore determines the over-all sample size for the experiments. Since N = 31 instead of N = 11, from Eq. (12. 19) one finds that the experiment will now detect a 6% difference in means ($\Delta\mu = 3$) while still maintaining a Type II error of 10%.

12. 2. 3 Acceptance Regions for Hypothesis Tests

(a) Chi-Square Test of Variances

For N = 31 sample records, the estimated sampling variance $\hat{\sigma}^2_{\nu_o}$ will be distributed about the true sampling variance $\sigma^2_{\nu_o}$ as follows.

$$\frac{\hat{\sigma}^2_{\nu_o}}{\sigma^2_{\nu_o}} \sim \frac{\chi^2(30)}{31}$$

where $\chi^2(30)$ is a chi-square distribution with $(N - 1) = 30$ degrees of freedom, and the symbol $\sim$ means "distributed as." From Table 5.3 of Ref. [1], for a 10% level of significance

$$\frac{\chi^2(30)_{0.05}}{31} = 1.412 \qquad \frac{\chi^2(30)_{0.95}}{31} = 0.5965$$

Then the acceptance region for testing the null hypothesis $H_o(\sigma^2)$ will be as follows:

$$0.5965 \leq \frac{\hat{\sigma}^2_{\nu_o}}{\sigma^2_{\nu_o}} \leq 1.412 \tag{12.23}$$

If the variance ratio falls within the limits stated in Eq. (12.23), $H_o(\sigma^2)$ will be accepted. If the variance ratio falls outside the limits stated, $H_o(\sigma^2)$ will be rejected.

(b) Normal Test of Mean Values

For N = 31 sample records, the estimated sampling mean $\hat{\mu}_{\nu_o}$ will be distributed about the true sampling mean μ_{ν_o} as follows.

$$(\hat{\mu}_{\nu_o} - \mu_{\nu_o}) \sim \text{Norm}\left(0\,;\, \frac{\sigma^2_{\nu_o}}{31}\right)$$

where Norm $(0\,;\, \sigma^2_{\nu_o} / 31)$ is a Normal distribution with a mean of zero and a variance of $\sigma^2_{\nu_o} / 31$. From Table 5.2 of Ref. [1], for a 10% level of significance

$$\text{Norm}\left(0\,;\, \frac{\sigma^2_{\nu_o}}{31}\right)_{0.05} = -\,\text{Norm}\left(0\,;\, \frac{\sigma^2_{\nu_o}}{31}\right)_{0.95} = \pm\, 1.645 \frac{\sigma_{\nu_o}}{\sqrt{31}}$$

Then the acceptance region for testing the null hypothesis $H_o(\mu)$ will be as follows.

$$-\frac{1.645}{\sqrt{31}}\,\sigma_{\nu_o} \leq (\hat{\mu}_{\nu_o} - \mu_{\nu_o}) \leq \frac{1.645}{\sqrt{31}}\,\sigma_{\nu_o} \tag{12.24}$$

If the mean difference falls within the limits stated in Eq. (12. 24), $H_o(\mu)$ will be accepted. If the mean difference falls outside the limits stated, $H_o(\mu)$ will be rejected.

(c) Student's "t" Test of Mean Values

Consider the case where the variance test (a) rejects the null hypothesis $H_o(\sigma^2)$. It is still of interest whether or not the estimated sampling mean is equivalent to the expected sampling mean for a random signal. However, in the absence of a known expected value for the sampling variance $\sigma^2_{\mathcal{V}_o}$, the estimated sampling variance $\hat{\sigma}^2_{\mathcal{V}_o}$ must be used to establish a region of acceptance for a hypothesis test.

For N = 31 sample records, the estimated sampling mean $\hat{\mu}_{\mathcal{V}_o}$ will be distributed about the true sampling mean $\mu_{\mathcal{V}_o}$ as follows.

$$(\hat{\mu}_{\mathcal{V}_o} - \mu_{\mathcal{V}_o}) \sim t\left(0\,;\frac{\hat{\sigma}^2_{\mathcal{V}_o}}{30}\right)$$

where $t(0\,;\hat{\sigma}^2_{\mathcal{V}_o}/30)$ is a "t" distribution with N - 1 = 30 degrees of freedom, a mean of zero, and a variance of $\hat{\sigma}^2_{\mathcal{V}_o}/30$. From Table 5. 4 of Ref. [1], for a 10% level of significance,

$$t_{0.05}\left(0\,;\frac{\hat{\sigma}^2_{\mathcal{V}_o}}{30}\right) = -t_{0.95}\left(0\,;\frac{\hat{\sigma}^2_{\mathcal{V}_o}}{30}\right) = \pm\frac{1.70}{\sqrt{30}}\,\hat{\sigma}_{\mathcal{V}_o}$$

Then, the acceptance region for testing the null hypothesis $H_o(\mu)$ will be as follows.

$$-\frac{1.70}{\sqrt{30}}\,\hat{\sigma}_{\mathcal{V}_o} \leq (\hat{\mu}_{\mathcal{V}_o} - \mu_{\mathcal{V}_o}) \leq \frac{1.70}{\sqrt{30}}\,\hat{\sigma}_{\mathcal{V}_o} \qquad (12.25)$$

If the mean difference falls within the limits stated in Eq. (12. 25), $H_o(\mu)$ will be accepted. If the mean difference falls outside the limits stated, $H_o(\mu)$ will be rejected.

12.2.4 Procedure for Experiments in Area A

A random signal with an approximately uniform power spectrum and Gaussian amplitude probability density function is sampled for three different frequency ranges, as follows.

A-1.	$f_a \approx 30$ cps,	$f_b \approx 2000$ cps
A-2.	$f_a \approx 30$ cps,	$f_b \approx 1000$ cps
A-3.	$f_a \approx 100$ cps,	$f_b \approx 600$ cps

The above frequency limits are only approximate because a more precise value for f_a and f_b must be calculated from the filter response characteristics, as is discussed in Section 12.3.2.

The frequency range for test A-1 represents the maximum frequency response range of the instrumentation employed. Test A-2 is conducted to permit statistical tests for a different range of frequencies. The frequency limits selected for test A-3 constitute the frequency range of the predominate excitation from large turbo-jet engines.

For each frequency range 31 sample records are collected, each representing at least 100 events. The number of zero crossings in each record is counted and analyzed as described in Sections 12.2.1 and 12.2.3.

12.2.5 Procedure for Experiments in Area B

A random signal with an approximately Gaussian amplitude probability density function and a frequency range of about 100 cps to 600 cps is sampled for four different power spectra. The power spectrum of the signal is altered by shaping filters which introduce sharp peaks into the spectrum. All four power spectra conditions are obtained using a filter with a very narrow bandwidth and a peak response of 10:1, corresponding to a power spectrum peak of 100:1. The sharp filtering is intended to simulate a resonant structural response. The four filter conditions are as follows.

- B-1. A single peak at 150 cps
- B-2. A single peak at 350 cps
- B-3. A single peak at 550 cps
- B-4. Three peaks at the three frequencies in B-1 through B-3

The detailed characteristics of the sharp filters used to obtain the peaks are presented in Section 12. 3. 3.

For each power spectrum condition, 31 sample records are collected, each representing at least 100 events. The number of zero crossings in each record is counted and analyzed as described in Sections 12. 2. 1 and 12. 2. 3.

12. 2. 6 Procedure for Experiments in Area C

A nonrandom signal is created by superimposing a sinusoidal component in a random signal with an approximately uniform power spectrum and Gaussian amplitude probability density function. Let f_o be the frequency of the sine wave component, let MS(sine) be the mean square value of the sine wave component, and let MS(random) be the mean square value of the random component.

The nonrandom signal is sampled for nine conditions as follows:

C-1 through C-4. $f_a \approx 30$ cps, $f_b \approx 1000$ cps

MS(sine) = 1/4 MS(random)

C-1. $f_o = 200$ cps

C-2. $f_o = 400$ cps

C-3. $f_o = 600$ cps

C-4. $f_o = 800$ cps

C-5 through C-9. $f_a \approx 100$ cps, $f_b \approx 600$ cps

MS(sine) = MS(random)

C-5. $f_o = 150$ cps

C-6. $f_o = 250$ cps

C-7. $f_o = 350$ cps

C-8. $f_o = 450$ cps

C-9. $f_o = 550$ cps

For each condition of nonrandomness, 31 sample records are collected, each representing at least 100 events. The number of zero crossings in each record is counted and analyzed as described in Sections 12. 2. 1 and 12. 2. 3.

12.3 INSTRUMENTATION

12.3.1 Instruments and Test Set-Up

The laboratory instruments employed for the experiments are listed in Table 12.1. All instruments were in current calibration at the time of the experiments. A block diagram for the test set-up is illustrated in Figure 12.3.

The random noise generator (Item A) is used as the source of a signal which is considered in these experiments to be truly random. The manufacturer's data indicates the instrument generates a random signal with an approximately Gaussian amplitude probability density function and a reasonably uniform power spectral density function for frequencies from 30 cps to 20 kcps. Actual measurements of the noise generator's characteristics are presented in Section 14.3. The voltmeter (Item B) is used to measure the rms voltage level of the random signal being delivered.

The sine wave generator (Item C) is used as the source of a signal which is known to be nonrandom. The voltmeter (Item D) is used to measure the rms voltage level of the sinusoidal signal being delivered. The frequency counter (Item E) is used to determine the frequency of the sinusoidal signal with an accuracy of ± 1 cps.

The sine wave-random noise mixer (Item F) is used to create a signal with controlled and defined nonrandom characteristics. The variable band pass filter (Item G) is used to band-limit the frequency range of the signal to be investigated. The characteristics and calibration of this filter are discussed in Section 12.3.2.

The shaping filters (Item H) are used to obtain a signal with a non-uniform power spectrum. Five peaks or notches may be inserted to simulate structural response modes. The characteristics of these filters are discussed in Section 12.3.3.

The voltmeter (Item I) is used to measure the rms voltage level of the signal to be investigated. The oscilloscope (Item J) is used to display the signal amplitude time history for qualitative observation.

The power amplifier (Item K) is used to obtain a sufficient signal current to operate the galvanometer oscillograph. The galvanometer oscillograph (Item L) is used to record amplitude time history samples of the signal to be investigated. The galvanometers employed have a uniform frequency response from 0 to 2000 cps.

Item	Description	Manufacturer	Model No.	Serial No.
A	Random Noise Generator	General Radio Company	1390A	937
B	True RMS Voltmeter	Ballantine Laboratories	320	581
C	Sine Wave Generator	Hewlett Packard	200 AB	5631
D	AC Voltmeter	Hewlett Packard	400 H	2075
E	Frequency Counter	Computer - Measurements Co.	200 B	6320
F	Sine Wave-Random Noise Mixer	Thompson Ramo Wooldridge Inc., RW Division	---	----
G	Variable Band Pass Filter	Krohn Hite Corp.	330-M	1484
H	Shaping Filters (bank of five peaks and notches)	Space Technology Laboratories	---	----
I	True RMS Voltmeter	Ballantine Laboratories	320	1149
J	Cathode Ray Oscillo-scope	Tektronix, Inc.	531	3190
K	Power Amplifier	McIntosh Company	A-116-B	12629
L	Galvanometer Oscillograph	Minneapolis-Honeywell	1012	307

Table 12.1 Instruments Employed for the Experiments

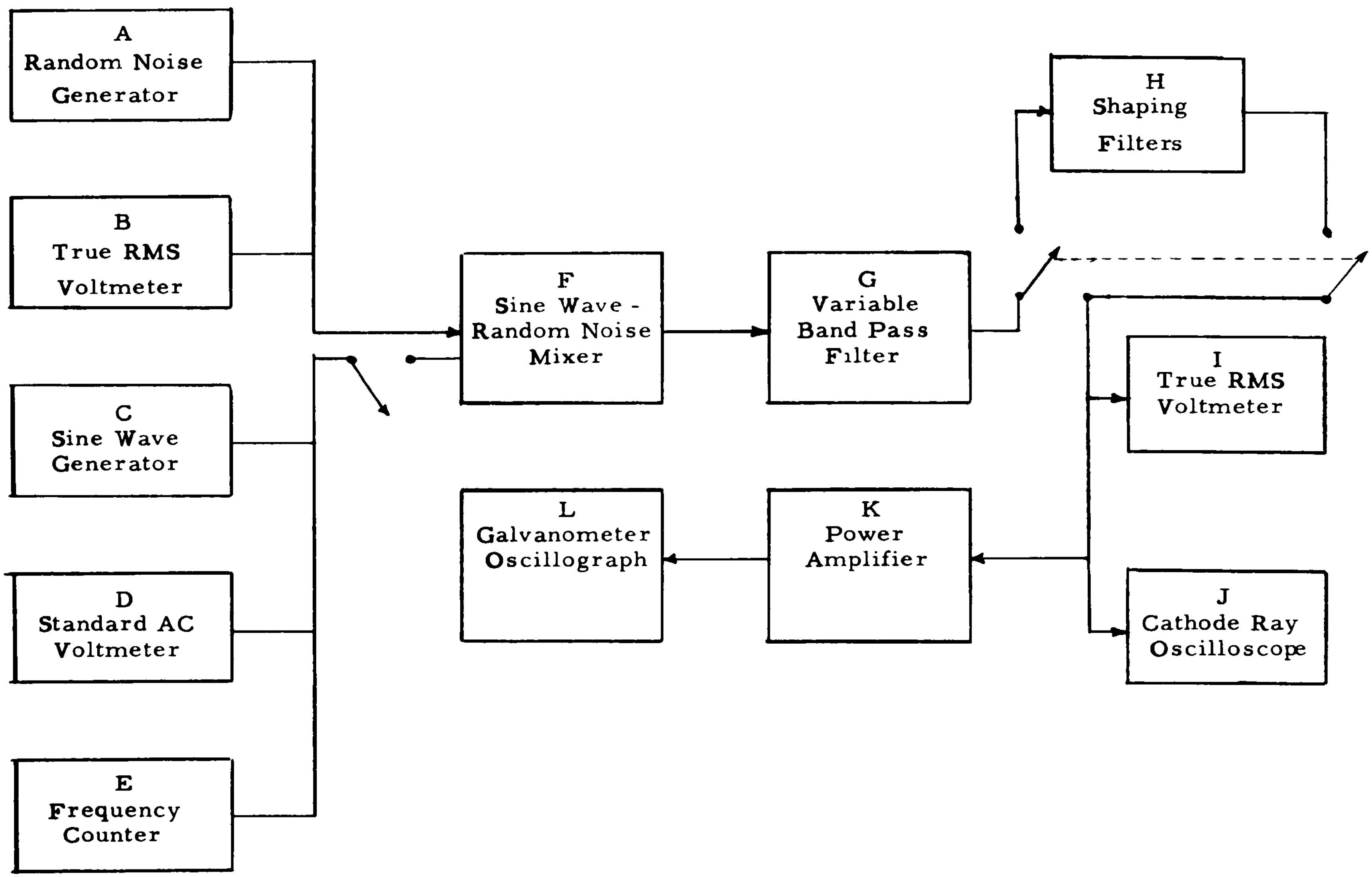


Figure 12.3 Block Diagram of Test Set-Up

12.3.2 Evaluation and Calibration of Variable Band Pass Filter

The number of events n represented by a sample record of length T, as given in Eq.(12.13) is directly proportional to the expected number of zero crossings per second $\bar{\nu}_0$, as defined by Eq.(12.11). The value of $\bar{\nu}_0$ is based upon the frequency bandwidth of the signal assuming infinitely sharp lower and upper frequency limits of f_a and f_b, respectively. Of course, ideally sharp frequency cut-offs are not physically realizable. However, given a filter with a frequency response function $H(f)$, the effective cut-off frequencies equivalent to f_a and f_b can be computed. From Refs. [1] and [2], the effective bandwidth B_N of a band pass filter as seen by a random signal with a uniform power spectrum is defined by

$$B_N = f_b - f_a = \int_0^{\infty} \left| \frac{H(f)}{H_{max}} \right|^2 df \qquad (12.26)$$

The term B_N is sometimes called the noise bandwidth of the filter. The integration in Eq(12.26) may be accomplished in two parts as follows.

$$B_N = f_b - f_a = \int_0^{f_c} \left| \frac{H(f)}{H_{max}} \right|^2 df + \int_{f_c}^{\infty} \left| \frac{H(f)}{H_{max}} \right|^2 df = B_a + B_b \qquad (12.27)$$

where f_c is some frequency at which $|H(f)| = |H_{max}|$. Then, B_a gives a noise bandwidth from f_c to an effective low frequency cut-off (f_a), and B_b gives a noise bandwidth from f_c to an effective high frequency cut-off (f_b). That is,

$$\begin{aligned} B_a &= f_c - f_a \\ B_b &= f_b - f_c \end{aligned} \qquad (12.28)$$

Rearranging the terms in Eq(12.28) and substituting from Eq. (12. 27), the effective frequency cut-offs for any band pass filter are given by,

$$f_a = f_c - \int_0^{f_c} \left|\frac{H(f)}{H_{max}}\right|^2 df$$

$$f_b = f_c + \int_{f_c}^{\infty} \left|\frac{H(f)}{H_{max}}\right|^2 df \qquad (12.29)$$

The cut-off characteristics of the band pass filter employed in these experiments have been carefully determined for the three frequency band-widths used. The results are presented in Figs. 12.4, 12.5, and 12.6. Both the quantities $|H(f)|$ and $|H(f)|^2$ are shown for $|H_{max}|$ = unity. The low frequency cut-off characteristics of the filter were studied only for the case were $f_a \approx 100$ cps.

The quantity $|H(f)|^2 (H_{max} = 1)$ was integrated by graphical techniques for each of the three high frequency cut-off conditions and the low frequency cut-off condition of $f_a \approx 100$ cps. The resulting effective cut-off frequencies, when compared to the half power point frequencies f_{hp}, are approximately 5% higher or lower than f_{hp} for the high frequency or low frequency cut-off, respectively. Hence, the effective cut-off frequencies for the variable band pass filter employed in these experiments are considered to be as follows.

$$f_a = 0.95\, f_{hp} \text{ (low frequency cut-off)}$$

$$f_b = 1.05\, f_{hp} \text{ (high frequency cut-off)}$$

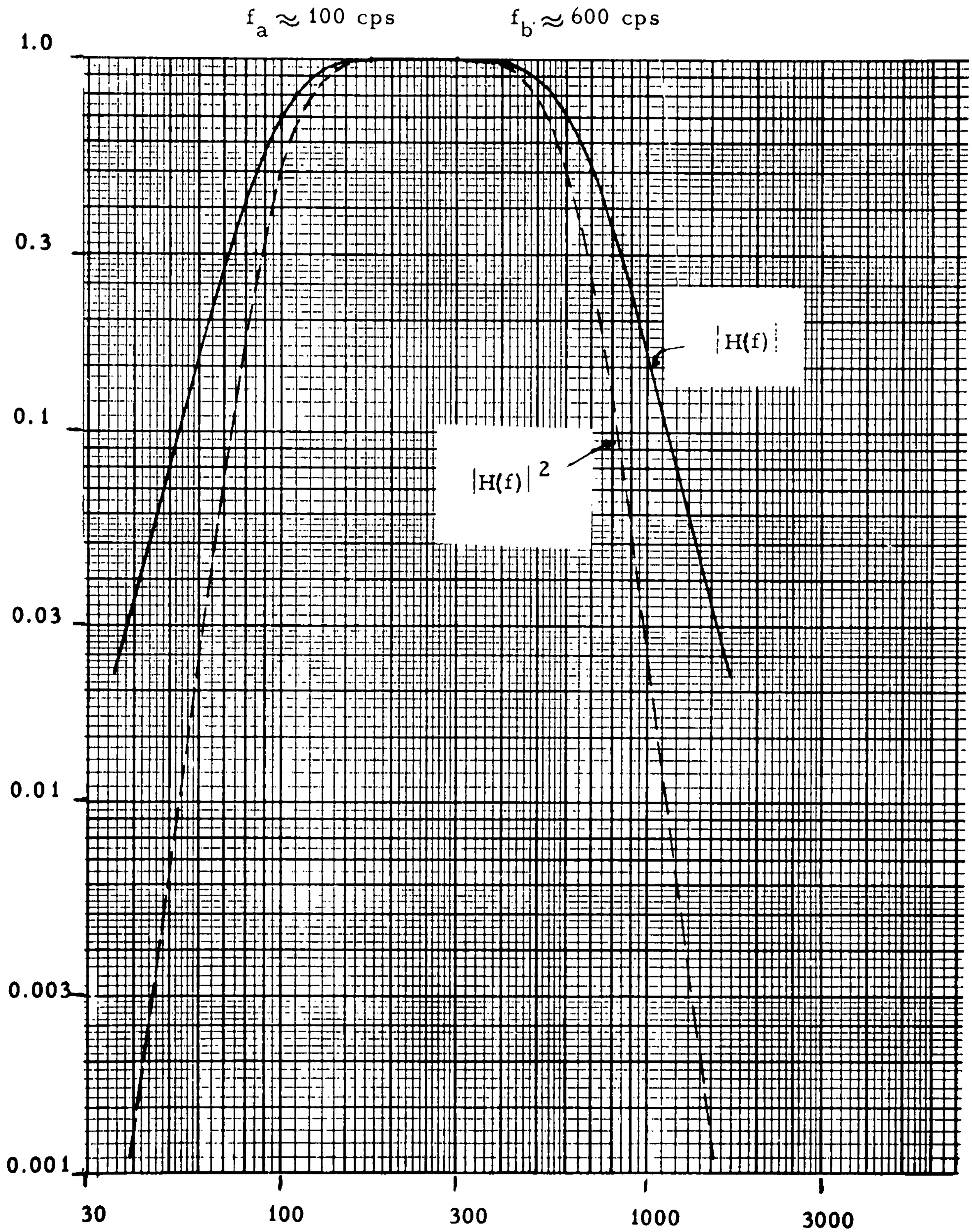


Figure 12.4 Variable Band Pass Filter Characteristic

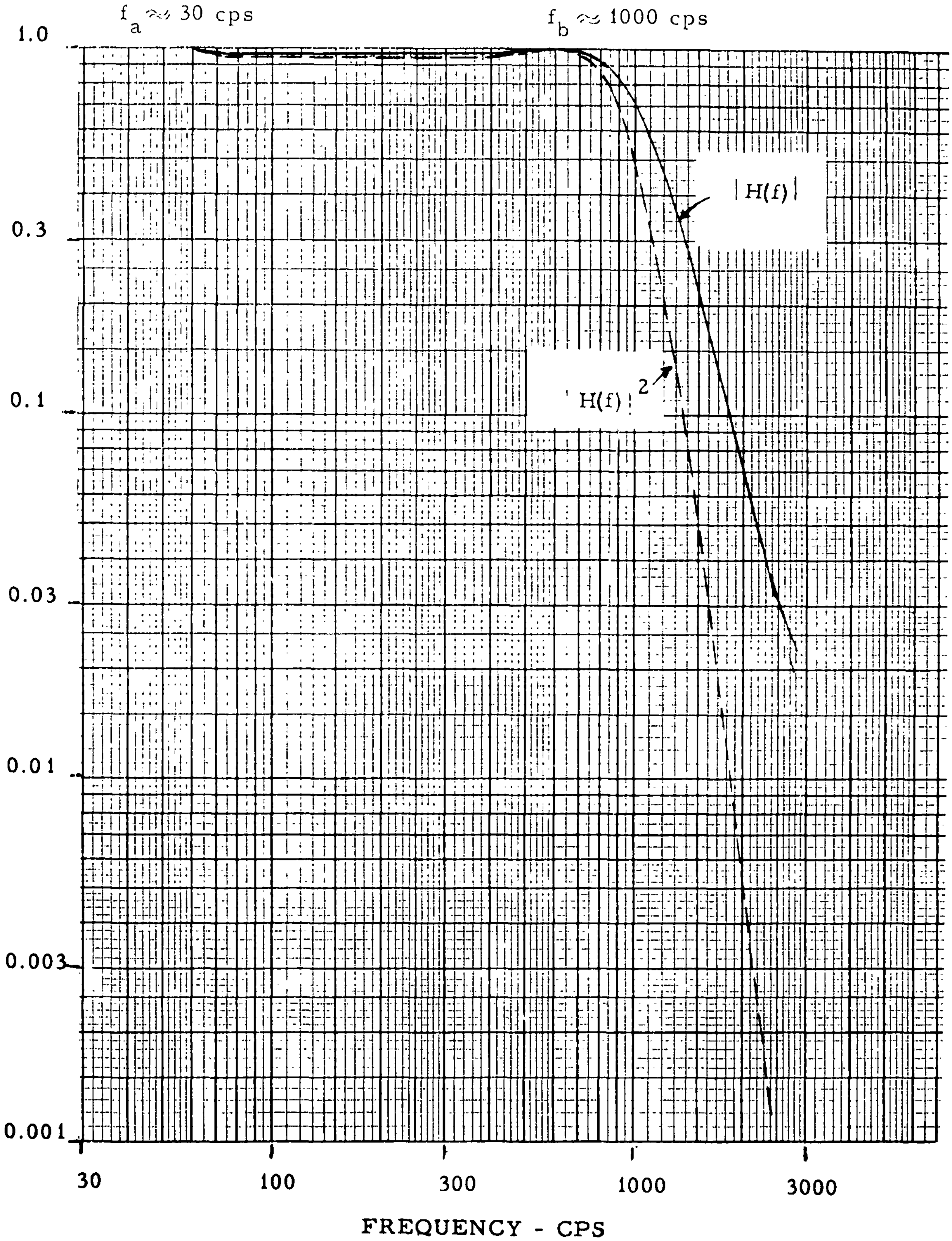


Figure 12.5 Variable Band Pass Filter Characteristic

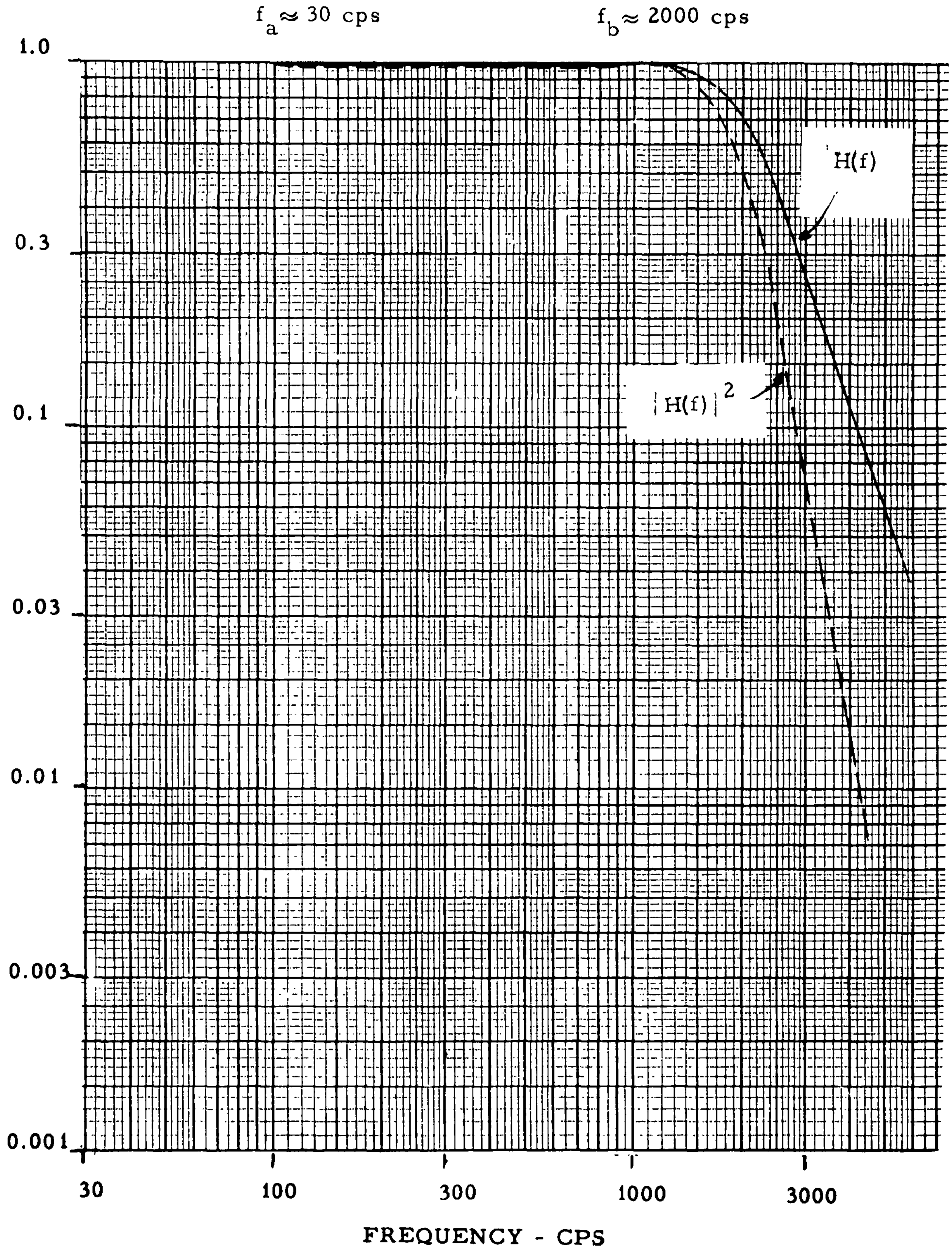


Figure 12.6 Variable Band Pass Filter Characteristic

12.3.3 Characteristics of Shaping Filters

The frequency response characteristics of the three narrow band peak filters, as employed for these experiments, are illustrated in Figure 12.7. The frequency cut-offs for the variable band pass filter are set for $f_a \approx 100$ cps and $f_b \approx 600$ cps. All three peaks are shown together. For the experiments, any two of the three peaks are bypassed as required for the procedure in Section 12.2.5.

12.3.4 Other Instrument Evaluations and Calibration Errors

There are, of course, numerous possible errors associated with the calibrations of all the instruments employed for the experiments. However, most of these calibration errors are not really significant in terms of the end data being sought; namely, the number of zero crossings by a random signal in a specific period of time. From Eqs. (12.11) and (12.13), it is clear that the major direct sources of error in the end data are as follows.

(a) error in the effective high frequency cut-off, f_b.

(b) error in the record length T.

(c) error in the interpretation of the number of zero crossings, ν_o.

The possible error in the value of f_b is the result of several factors including the determination of the half power point of the high frequency cut-off, the calculation of the effective cut-off frequency as compared to the half power point, and drift in the filter characteristics. Referring to Fig. 12.3, the half power point is determined by reading the frequency from Item E when the filter response is 3 db down from the maximum response, as read from Item I, for a sine wave applied at a constant input voltage, as read from Item D. The indirect sources of error are then the lack of a perfect frequency response in the two voltmeters (Items D and I),

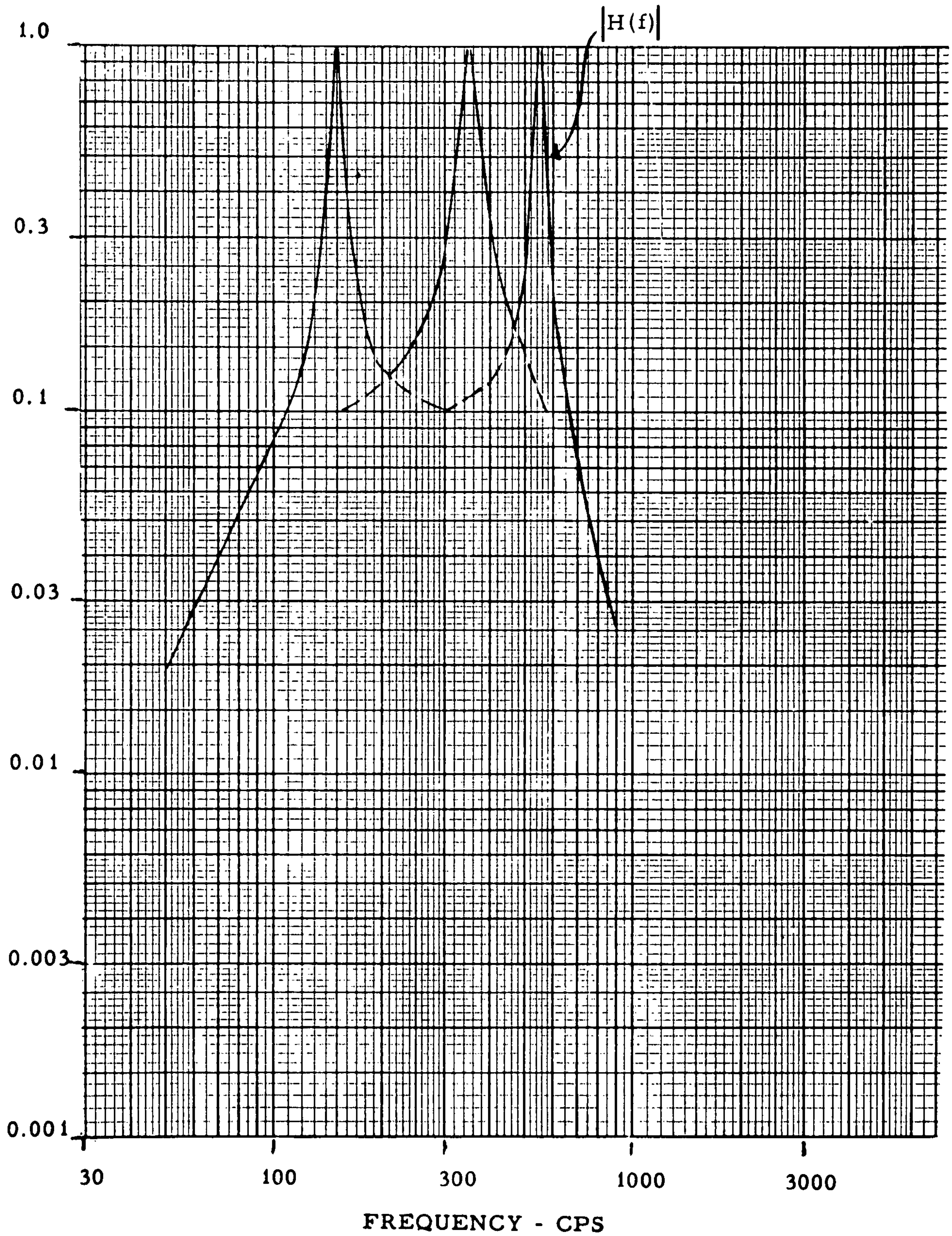


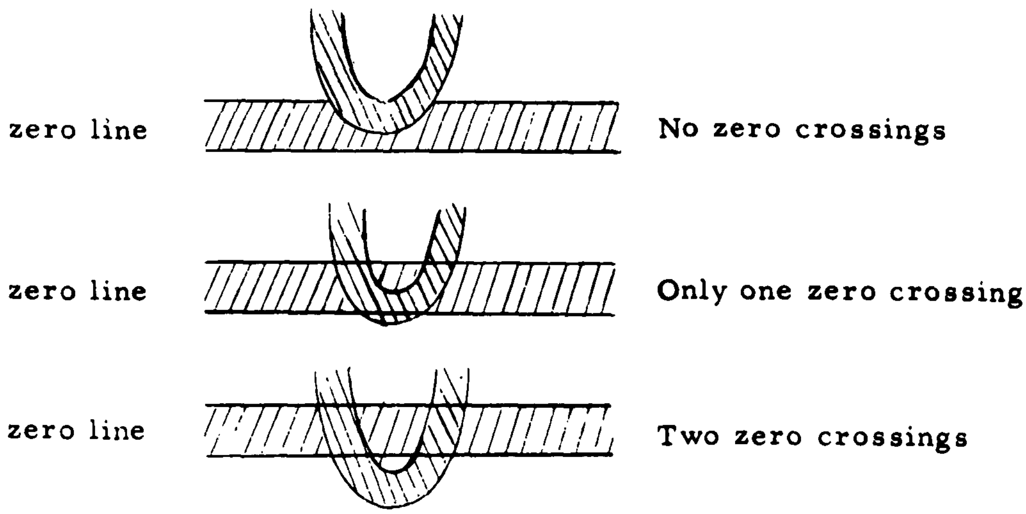
Figure 12.7 Narrow Band Shaping Filter Characteristics

the ± 1 cps accuracy of the frequency counter (Item D), and observational errors. It is believed that the total effect of these factors on the determination of the half power point frequency is less than ± 1% for these experiments. The half power point is carefully checked immediately before each experiment. Since a single record run is never greater than about three seconds, the effect of filter drift is considered negligible. The computation of the effective cut-off frequency for any given half power point is assessed to be within ± 2%. Thus, the maximum error in a value of f_b is believed to be less than ±3%.

The possible error in the record length T is a function of the timing lines generated by the galvanometer oscillograph (Item L). The accuracy of the timing lines have been checked by recording a sine wave with a frequency of 1000 cps determined to ± 1 cps using Item E. The results indicate the error in the timing lines is insignificant.

The possible error in the interpretation of the number of zero crossings ν_o is the result of two factors. First, the correct identification of the zero line or mean value of the signal, and second, the correct identification of a crossing. The mean value of the signal is automatically rendered zero by the filtering conditions as well as the lack of DC response capabilities in other instruments. By carefully setting the galvanometer on the zero base line of the record before each experiment, the error in identifying the zero line becomes insignificant. However, the zero line on the record, as well as the signal trace, has a finite width or thickness. This introduces the problem of interpreting a zero crossing when a signal peak occurs near the zero line. The following definitions are employed in

in these experiments. Note that the width of the trace is exaggerated for clarity.



There are other less tangible errors associated with the experiments that have not been mentioned. The most significant of these is the possible effect of the filter skirts. As the bandwidth of the variable band pass filter is made narrower, the effect of the skirts on the over-all filter band pass characteristics is increased. In other words, if f_a is a substantial fraction of f_b, a large part of the filter transmission of a random signal is occurring in the skirts. It is difficult to assess the error that might be introduced by this effect, but the possibility of such an error should be noted.

Clipping of the random signal by the instruments is another source of possible error. To minimize the possible effects of clipping, all experiments are conducted with a signal level giving a crest factor of at least 4. That is, the rms voltage of the signal was established at less than 1/4 the voltage level where clipping would occur in any of the instruments.

12.4 RESULTS OF EXPERIMENTS

12.4.1 Zero Crossing Data

The number of zero crossings counted in each of the 31 sample records gathered for each experiment are presented in Tables 12.2 and 12.3. The results of experiments with a random signal are covered in Table 12.2, and the results of experiments with a nonrandom signal are covered in Table 12.3. The test designations refer to the test procedures defined in Section 12.2.

Sample Number	Number of Zero Crossings ν_0 For Sample Records of Length T Seconds Test Procedure						
	A-1 T=0.03	A-2 T=0.05	A-3 T=0.1	B-1 T=0.1	B-2 T=0.1	B-3 T=0.1	B-4 T=0.1
1	65	61	77	68	74	107	91
2	68	69	70	45	75	104	87
3	59	57	76	70	71	109	95
4	79	60	81	54	70	107	85
5	73	61	91	36	77	101	72
6	81	78	78	63	71	108	82
7	70	68	79	37	80	108	101
8	77	67	74	74	77	110	97
9	72	59	85	57	74	106	79
10	78	64	80	48	81	108	85
11	65	63	94	53	73	105	89
12	76	55	83	55	72	109	81
13	76	68	80	59	74	105	95
14	69	65	83	49	76	107	85
15	66	62	89	39	72	107	88
16	77	62	74	53	76	110	93
17	76	71	75	56	75	107	99
18	70	60	87	59	73	109	104
19	73	62	93	59	68	110	91
20	69	62	64	59	76	107	91
21	73	55	88	56	75	108	98
22	80	60	87	51	73	109	92
23	72	68	79	36	81	108	97
24	82	64	83	73	72	110	82
25	68	60	80	58	70	107	90
26	74	68	84	51	71	106	98
27	83	66	91	57	71	101	82
28	59	66	85	57	73	103	90
29	74	55	78	68	71	105	98
30	64	55	86	57	72	109	104
31	67	59	81	43	73	105	98

Table 12.2 **Results of Experiments with a Random Signal**

Sample Number	Number of Zero Crossings ν_0 For Sample Records of Length T Seconds Test Procedure								
	C-1 T=0.05	C-2 T=0.05	C-3 T=0.05	C-4 T=0.05	C-5 T=0.10	C-6 T=0.10	C-7 T=0.10	C-8 T=0.10	C-9 T=0.10
1	60	62	64	68	51	68	76	89	94
2	65	71	74	69	55	68	82	89	94
3	55	67	68	73	55	63	80	91	102
4	60	64	66	69	58	71	78	90	95
5	53	58	76	72	68	69	75	85	98
6	61	67	62	66	48	69	79	84	103
7	69	58	67	71	64	66	79	84	102
8	61	69	72	68	61	69	80	91	99
9	58	70	58	70	61	70	76	86	104
10	59	68	70	63	48	74	80	92	108
11	47	50	64	73	63	68	75	90	100
12	54	65	65	66	58	76	90	85	105
13	56	72	62	72	56	70	82	84	102
14	57	63	68	66	58	63	83	91	105
15	58	58	63	74	65	66	78	92	103
16	56	73	60	84	60	67	73	94	106
17	59	60	70	76	66	65	77	92	108
18	48	70	65	66	56	68	78	88	104
19	62	59	70	58	54	70	77	82	99
20	61	63	61	68	68	56	74	90	105
21	61	69	65	81	70	62	77	86	106
22	43	59	70	73	60	68	71	87	100
23	67	68	70	74	57	67	75	84	109
24	57	68	73	73	58	63	75	92	102
25	67	64	71	66	60	63	80	87	108
26	63	64	70	76	62	74	73	83	102
27	61	67	78	71	61	67	81	89	104
28	65	62	62	73	56	67	75	89	107
29	58	69	60	74	65	61	73	91	100
30	62	55	70	63	71	71	80	84	109
31	54	61	66	72	56	69	78	89	101

Table 12.3 Results of Experiments with a Nonrandom Signal

The effective low and high frequency band pass limits for the experiments are presented in Table 12.4. The effective cutoff frequencies are determined as discussed in Section 12.3.2.

Experiment by Test Procedure (Section 12. 2)	Effective Low Frequency Cut-Off f_a (cps)	Effective High Frequency Cut-Off f_b (cps)
A-1	28.5	2100
A-2	28.5	1050
A-3	95	630
B-1 through B-4	95	630
C-1 through C-4	28.5	1050
C-5 through C-9	95	630

Table 12.4 **Effective Frequency Band Pass Limits**

12.4.2 Statistical Estimates from Zero Crossing Data

From the zero crossing data presented in Section 12. 4. 1, the estimated mean value $\hat{\mu}_{\nu_o}$ and the estimated variance $\hat{\sigma}^2_{\nu_o}$ of the sampling distribution for each experiment are determined using Eqs.(12.14) and (12.15). The estimates are presented in Table 12. 5.

Description of Experiment	Effective Frequency Range (cps)	Test Procedure (Section 12. 2)	Mean Value $\hat{\mu}_{\nu_o}$	Variance $\hat{\sigma}^2_{\nu_o}$
random signal with an approximately uniform power spectrum	28.5 to 2100	A-1	72.10	38.03
	28.5 to 1050	A-2	62.90	27.00
	95 to 630	A-3	81.77	44.23
random signal with a non-uniform power spectrum	95 to 630	B-1	54.84	99.16
		B-2	73.77	9.65
		B-3	106.94	5.68
		B-4	90.94	57.87
nonrandom signal (sine wave plus random noise)	28.5 to 1050	C-1	58.61	32.35
		C-2	64.29	28.52
		C-3	67.10	23.58
		C-4	70.58	26.52
	95 to 630	C-5	59.65	32.49
		C-6	67.35	16.35
		C-7	77.74	13.74
		C-8	88.06	10.38
		C-9	102.71	16.52

Table 12. 5 **Estimated Mean and Variance of Sampling Distribution**

12.4.3 Expected Values for Random Signal Zero Crossings

For sample records gathered from a random signal, the expected values of the mean and variance of the sampling distribution, as predicted by run theory, are computed using Eqs. (12.7), (12.8), (12.11), and (12.13). These expected values for the effective frequency bandwidths used in the experiments are presented in Table 12.6.

Experiment by Test Procedure (Section 12.2)	f_a (cps)	f_b (cps)	$\bar{\nu}_o$ Eq. (12.11)	n Eq. (12.13)	μ_{ν_o} Eq. (12.7)	$\sigma^2_{\nu_o}$ Eq. (12.8)	σ_{ν_o}
A-1	28.5	2100	2441.49	146.49	73.24	36.37	6.03
A-2	28.5	1050	1229.22	122.92	61.46	30.48	5.52
A-3	95	630	788.00	157.60	78.80	39.14	6.26
B-1 through B-4	same as A-3						
C-1 through C-4	same as A-2						
C-5 through C-9	same as A-3						

Table 12.6 Expected Values for Mean and Variance of Zero Crossings

12.4.4 Statistical Testing of Zero Crossing Data

Let the following null hypotheses be established.

$$H_o(\sigma^2) \quad : \quad \hat{\sigma}^2_{\nu_o} = \sigma^2_{\nu_o}$$

$$H_o(\mu) \quad : \quad \hat{\mu}_{\nu_o} = \mu_{\nu_o}$$

The above null hypotheses are now tested for the data presented in Tables 12.5 and 12.6, by the procedures outlined in Section 12.2.3. The results for tests of $H_o(\sigma^2)$ are included in Table 12.7, and the results for tests of $H_o(\mu)$ are presented in Table 12.8.

Experiment by Test Procedure (Section 12. 2)	$\frac{\hat{\sigma}^2_{\nu_o}}{\sigma^2_{\nu_o}}$	Region of Acceptance For Test of $H_o(\sigma)$ from Eq. (12. 23)	Result of Test
A-1	1. 046	0. 5965 to 1. 412	accepted
A-2	0. 886		accepted
A-3	1. 130		accepted
B-1	2. 533		rejected
B-2	0. 246		rejected
B-3	0. 145		rejected
B-4	1. 478		rejected
C-1	1. 061		accepted
C-2	0. 936		accepted
C-3	0. 774		accepted
C-4	0. 870		accepted
C-5	0. 830		accepted
C-6	0. 418		rejected
C-7	0. 351		rejected
C-8	0. 265		rejected
C-9	0. 422		rejected

Table 12.7 Tests for Equivalence of Variances

Experiment by Test Procedure (Section 12. 2)	$\hat{\mu}_{\nu_o} - \mu_{\nu_o}$	Region of Acceptance For Test of $H_o(\mu)$ from Eq. (12.24) or (12. 25)	Result of Test
A-1	-1. 14	-1. 78 to 1. 78	accepted
A-2	1. 44	-1. 63 to 1. 63	accepted
A-3	2. 97	-1. 85 to 1. 85	rejected
B-1	-23. 96	-3. 09 to 3. 09	rejected *
B-2	-5. 03	-0. 96 to 0. 96	rejected *
B-3	28. 14	-0. 74 to 0. 74	rejected *
B-4	12. 14	-2. 36 to 2. 36	rejected *
C-1	-2. 85	-1. 63 to 1. 63	rejected
C-2	2. 83	-1. 63 to 1. 63	rejected
C-3	5. 64	-1. 63 to 1. 63	rejected
C-4	9. 12	-1. 63 to 1. 63	rejected
C-5	-19. 15	-1. 85 to 1. 85	rejected
C-6	-11. 45	-1. 25 to 1. 25	rejected *
C-7	-1. 06	-1. 15 to 1. 15	accepted *
C-8	9. 26	-1. 00 to 1. 00	rejected *
C-9	23. 91	-1. 26 to 1. 26	rejected *

*null hypothesis $H_o(\mu)$ tested by student's "t" test.

Table 12.8 Tests for Equivalence of Means

12.5 DISCUSSION OF RESULTS

12.5.1 Results for Experiments in Area A

The experiments in Area A deal with sample records from a signal which is considered truly random, and which has an approximately uniform power spectrum and Gaussian amplitude probability density function. Then, the distribution for the number of zero crossings in a sample record should have a mean and variance as theoretically predicted by Eqs.(12.7) and (12.8). If Eqs. (12.7) and (12.8) are valid, the null hypothesis tests should accept the estimated sampling mean and variance for all data in Area A as being statistically equivalent to the theoretical values. The results of the null hypothesis tests in Area A are summarized in Table 12.9.

Experiment by Test Procedure (Section 12.2)	Effective Frequency Range (cps)	Results of Null Hypothesis Tests	
		$H_o(\sigma^2)$	$H_o(\mu)$
A-1	28.5 to 2100	accepted	accepted
A-2	28.5 to 1050	accepted	accepted
A-3	95 to 630	accepted	rejected

Table 12.9 Summary of Results for Experiments in Area A

The experimentally estimated and theoretical expected sampling variances are accepted as equivalent for all three experiments performed. The estimated and expected sampling means are accepted as equivalent for two of the three experiments, but rejected for the experiment where the signal frequency range is 95 to 630 cps.

From Table 12.8, it is seen that the differences, $\hat{\mu}_{\nu_o} - \mu_{\nu_o}$, for Test Procedure A-3 is not far outside the region of acceptance. The rejection could be a Type I error (rejection of the hypothesis when in fact it is true), which is to be expected in one of every ten experiments conducted in Area A. It could also be the result of a bias error due to the relatively narrow frequency range for the experiment, as discussed in Section 12.3.4. Because the equivalence of variances and means are

accepted for all other tests in Area A, and in light of the above mentioned possible errors, it is believed a strong qualitative argument exists to neglect the single rejection in Table 12.9. Thus, the results of experiments in Area A are considered as confirmation of the validity of Eqs. (12.7) and (12.8).

12.5.2 Results for Experiments in Area B

The experiments in Area B deal with sample records from a signal which is considered truly random and has an approximately Gaussian amplitude probability density function, but does not have a uniform power spectrum. These experiments are conducted to determine if Eqs. (12.7) and (12.8) are acceptable approximations for the sampling distribution mean and variance of the zero crossings of a random signal that does not have a uniform power spectrum.

From Section 12.4.4, it is seen that the null hypothesis of equivalence for the experimentally estimated and theoretical expected sampling mean and variance are rejected for every experiment conducted. Clearly, Eqs. (12.7) and (12.8) are not valid expressions for the expected sampling mean and variance when the random signal being sampled does not have a uniform power spectrum.

12.5.3 Results for Experiments in Area C

The experiments in Area C deal with sample records from a signal which is known to be nonrandom. The results of the null hypothesis tests are presented in Table 12.10.

Test Procedures C-1 through C-4 involve a sine wave component with a mean square value equal to only 1/4 the mean square value of the background random signal. For these experiments, the estimated sampling variances are accepted as equivalent to the theoretical expected sampling variance for the random signal alone. It appears that the sine wave component is not sufficiently intense here to significantly influence the sampling variance. However, the estimated sampling means are not accepted as equivalent to the expected sampling mean for the random signal alone.

Experiment by Test Procedure (Section 12.2)	Effective Frequency Range of Random Components (cps)	Frequency of Sine Wave, f_o (cps)	$\frac{MS(sine)}{MS(random)}$	Results of Null Hypothesis Tests	
				$H_o(\sigma^2)$	$H_o(\mu)$
C-1	28.5 to 1050	200	1/4	accepted	rejected
C-2	28.5 to 1050	400	1/4	accepted	rejected
C-3	28.5 to 1050	600	1/4	accepted	rejected
C-4	28.5 to 1050	800	1/4	accepted	rejected
C-5	95 to 630	150	1	accepted	rejected
C-6	95 to 630	250	1	rejected	rejected
C-7	95 to 630	350	1	rejected	accepted
C-8	95 to 630	450	1	rejected	rejected
C-9	95 to 630	550	1	rejected	rejected

Table 12.10 Summary of Results for Experiments in Area C

Test Procedures C-5 through C-9 involve a sine wave component with a mean square value equal to the mean square value of the background random signal. Here the sine wave component is sufficiently intense to influence the sampling variance. The hypothesis of equivalent variances is rejected in four of the five experiments. Likewise, the hypothesis of equivalent means is rejected in all but one case, Test Procedure C-7. For this experiment, it appears the frequency of the sine wave (f_o = 350 cps) is just right to produce, in conjunction with the random component, approximately the same number of zero crossings that the random signal alone would produce.

12.6 CONCLUSIONS

The application of statistical run theory to the prediction of sampling distributions for random signal zero crossings has been studied experimentally. The results confirm the validity of basic theory. The theory correctly predicts the sampling mean and variance for zero crossings of signals with a reasonably uniform power spectrum.

The presence of a sine wave in the random signal significantly alters the resulting zero crossings, which indicates possible applications of the run theory to a test for randomness. However, the lack of a uniform power spectrum also alters the resulting zero crossings from theoretical values. These matters are considered in terms of a test for randomness in Section 15.

12.7 REFERENCES

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13. UNCERTAINTY OF POWER SPECTRA (MEAN SQUARE) ESTIMATES

13.1 THEORY OF POWER SPECTRAL DENSITY ESTIMATION

13.1.1 Review of Mathematical Relationships

Consider a stationary random signal, $x(t)$. Assume the signal is sharply band-limited between any two frequencies, f_a cps and f_b cps, giving an ideal bandwidth of $B = (f_b - f_a)$ cps. The mean square value of $x(t)$ within the bandwidth B may be estimated by

$$\overline{x^2}(B) = \frac{1}{T}\int_0^T x_B^2(t)\,dt \qquad (13.1)$$

where T is the record length (total time of observation). The estimated mean square value in Eq. (13.1) will approach the expected mean square value $E\left[\overline{x^2}\right]$ (true mean square value of the signal which hypothetically exists over all time) as the record length T approaches infinity. That is,

$$E\left[\overline{x^2}(B)\right] = \lim_{T\to\infty} \frac{1}{T}\int_0^T x_B^2(t)\,dt \qquad (13.2)$$

Furthermore, if the mean value of $x(t)$ is zero (no DC component is present or the bandwidth B does not include zero frequency), the expected mean square value is equal to the variance of $x(t)$, denoted by $\sigma_x^2(B)$. That is, assuming a zero mean value for $x(t)$,

$$E\left[\overline{x^2}(B)\right] = \sigma_x^2(B) \qquad (13.3)$$

The expected or true mean square value for the signal $x(t)$ for any bandwidth B will henceforth be denoted by $\sigma_x^2(B)$, as defined in Eq. (13.3).

The power spectral density function is given by

$$G(f) = \lim_{B \to 0} \frac{\sigma_x^2(B)}{B} = \lim_{B \to 0} \lim_{T \to \infty} \frac{1}{BT} \int_0^T x_B^2(t)\, dt \tag{13.4}$$

where f is the center frequency of the bandwidth B. The power spectral density function $G(f)$ then defines the mean square value (relative power) of a signal $x(t)$ between any two frequency limits, f_1 and f_2, as follows.

$$\sigma_x^2(f_1, f_2) = \int_{f_1}^{f_2} G(f)\, df \tag{13.5}$$

The power spectral density function for stationary random signals may also be defined by the Fourier Transform of the correlation function (see Ref. [1], Section 4.4.2). However, the definition presented in Eq. (13.4) better illustrates the physical operations required to estimate power spectra with analog instruments.

From Eq. (13.4), the procedure of taking the limit as B approaches zero is beyond the capabilities of physical instruments and, of course, the record length T must always be finite. However, if B is small, an estimate for the power spectral density at any frequency f for a sample record length T is given by

$$\hat{G}(f) = \frac{\overline{x^2}(B)}{B} = \frac{1}{BT} \int_0^T x_B^2(t)\, dt \tag{13.6}$$

Equation (13.6) defines the basic operations which an analog instrument must accomplish to estimate the power spectrum of a random signal from a sample record of length T.

1. Filtering of the signal by some narrow frequency window having a bandwidth B.

2. Squaring of the filtered signal amplitudes and integration of the squared amplitudes over the record length T.

3. Division by the record length T.

4. Division by the frequency window bandwidth B.

Of course, the center frequency of the narrow bandwidth B would have to be variable to cover the entire range of frequencies under consideration. The actual circuit techniques employed by analog instruments to accomplish the above functions are discussed in Ref. [1], Sections 7.3 and 7.4.

13.1.2 Theoretical Evaluation of Estimation Uncertainty

The statistical uncertainty associated with power spectral density estimates is developed in Ref. [1], Section 4.8. From those results, the uncertainty for a power spectral density estimate at any frequency f cps for a filter bandwidth of B cps and a record length of T seconds is given by

$$E\left[\hat{G}(f) - G(f)\right]^2 \approx \frac{G^2(f)}{BT} + \frac{B^4}{576}\left[G''(f)\right]^2 \tag{13.7}$$

The first term in Eq. (13.7) is an expression of variability and the second term is an expression of bias. The bias term will clearly be negligible except for those cases where the slope of the power spectrum is changing rapidly within the bandwidth B. For most cases, the estimation uncertainty may be considered to be the first term of Eq. (13.7), which is conveniently written in the form of a dimensionless variance as follows.

$$\epsilon^2 = \frac{\sigma^2_{\hat{G}(f)}}{G^2(f)} \approx \frac{1}{BT} \tag{13.8}$$

Note that B is an ideal frequency window; that is, a bandwidth with infinitely sharp cutoffs.

The results of Eq. (13.8) may be arrived at by a procedure different from the derivation presented in Ref. [1]. From Ref. [2], the equivalent number of events n represented by a continuous white noise signal record of length T with a bandwidth B is given by

$$n = 2BT \tag{13.9}$$

A power spectral density estimate is effectively a mean square value or variance estimate. Hence, if the signal $x(t)$ is assumed to have an approximately Gaussian probability density function, the sampling distribution for an estimate $\hat{G}(f)$ will be

$$\hat{G}(f) \sim \frac{G(f)\,\chi^2}{(n-1)} \tag{13.10}$$

where "$\sim$" means "distributed as " and χ^2 is a chi-squared distribution with (n - 1) degrees of freedom. Noting that all terms in Eq. (13.10) are expected or exact values except for χ^2, the variance of $\hat{G}(f)$ is given by

$$\sigma^2_{\hat{G}(f)} = \mathrm{Var}\left[\frac{G(f)\,\chi^2}{(n-1)}\right] = \left[\frac{G(f)}{(n-1)}\right]^2 \mathrm{Var}\,\chi^2 \tag{13.11}$$

However, the variance of a chi-squared distribution is equal to twice the degrees of freedom, or $\mathrm{Var}\,\chi^2 = 2(n-1)$. Then,

$$\sigma^2_{\hat{G}(f)} = \frac{2G^2(f)(n-1)}{(n-1)^2} = \frac{2G^2(f)}{n-1} \tag{13.12}$$

Substituting Eq. (13.9) into Eq. (13.12), and assuming $n \gg 1$, it follows that

$$\epsilon^2 = \frac{\sigma^2_{\hat{G}(f)}}{G^2(f)} \approx \frac{1}{BT} \tag{13.13}$$

which is the same result presented in Eq. (13.8).

13.1.3 Application of Estimation Uncertainty

The positive square root of Eq. (13.8) or Eq. (13.13) defines the normalized standard error of an estimate $\hat{G}(f)$ as follows.

$$\epsilon = \frac{\sigma_{\hat{G}(f)}}{G(f)} \approx \frac{1}{\sqrt{BT}} \tag{13.14}$$

The term ϵ is simply the standard deviation of the estimate expressed as a fractional portion of the power spectral density being measured. For example, assume the power spectrum of a signal is measured by analyzing a sample record with a BT product of 100. The resulting estimate $\hat{G}(f)$ at any given frequency f will have a standard deviation of 10% of the true power spectral density at that frequency. A plot of the uncertainty ϵ versus the BT product, as given by Eq. (13.14), is shown in Figure 13.1

The uncertainty of $\hat{G}(f)$ may also be expressed in terms of confidence intervals. As noted in Section 13.1.2, Eq. (13.10), if the signal $x(t)$ has an approximately Gaussian probability density function, the sampling distribution for $\hat{G}(f)$ at any frequency f will be as follows.

$$\frac{\hat{G}(f)}{G(f)} \sim \frac{\chi^2}{(n-1)} \tag{13.15}$$

Given an estimate $\hat{G}(f)$ obtained from a sample record with $n = 2BT$ events, a $(1 - \alpha)$ confidence interval for the true power spectral density at any given frequency will be

$$\frac{(n-1)\hat{G}(f)}{\chi^2_{\alpha/2}} \leq G(f) \leq \frac{(n-1)\hat{G}(f)}{\chi^2_{1-\alpha/2}} \tag{13.16}$$

where $\chi^2_{1-\alpha/2}$ and $\chi^2_{\alpha/2}$ have $(n-1)$ degrees of freedom. See Ref. [1], Sections 7.3 and 7.4 for examples.

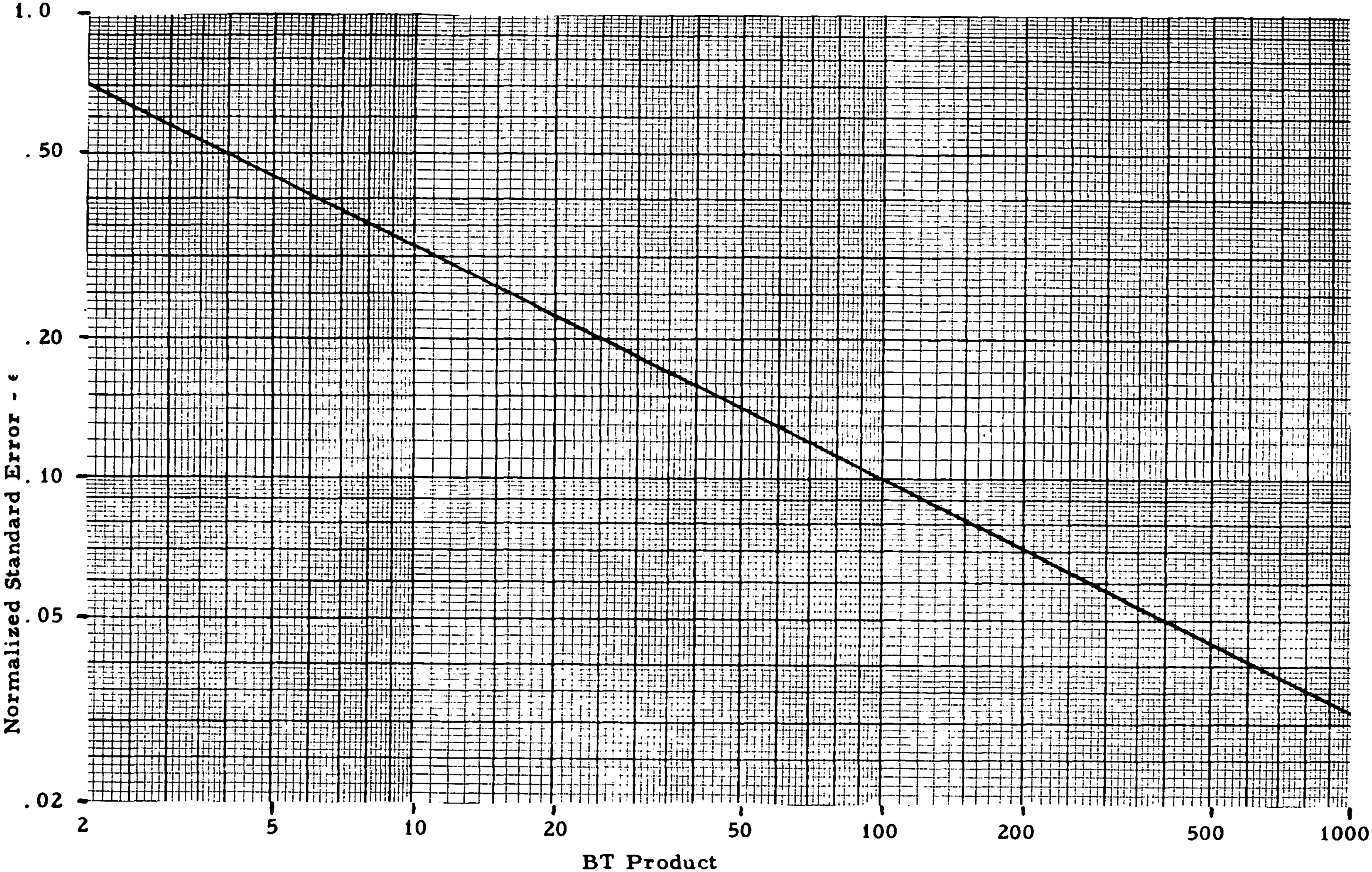

Figure 13.1 Estimation Uncertainty Versus BT Product

It should be noted that $n = 2BT$ is usually in practice a number much greater than one. As a result, it is common to assume $n \approx (n-1)$ and to use confidence intervals based upon χ^2 with n degrees of freedom.

Referring to Eq. (13.14), the bandwidth B is assumed to be ideal with infinitely sharp cutoffs. For physical applications where the narrow bandpass filter does not have ideal cutoffs, an equivalent ideal bandwidth (the noise bandwidth B_N) must be computed from the actual filter characteristics as follows.

$$B = B_N = \int_0^{\infty} \left| \frac{H(f)}{H_{max}} \right|^2 dt \tag{13.17}$$

where $H(f)$ is the frequency response function of the narrow bandpass filter. Then, for practical applications, the bandwidth B in Eq. (13.14) should be determined using Eq. (13.17). However, if the narrow bandpass filter of the analyzer has a cutoff of at least 60 db per octave, B may be considered as the bandwidth between half power points with negligible error. This is usually true for commercial power spectral density analyzers.

Again referring to Eq. (13.14), the record length T is the total length of the sample amplitude time history available for analysis. It is assumed that a power spectral density estimate for each filter center frequency is obtained by averaging over the entire available length of data. In other words, in Eq. (13.14), the averaging time and the record length are ideally the same value, T.

Consider now the cases where the averaging time is not equal to the record length T. Let T_1 be the averaging time. If T_1 is less

than T, the normalized standard error ϵ will be determined by the value of T_1. That is,

$$\epsilon \approx \frac{1}{\sqrt{BT_1}} \qquad T_1 \leq T \tag{13.18}$$

However, if T_1 is greater than T, the value for ϵ is not reduced below that value obtained using T in Eq. (13.14). That is,

$$\epsilon \approx \frac{1}{\sqrt{BT}} \qquad T < T_1 \tag{13.19}$$

In other words, there is only so much data available for analysis; namely, T seconds of data. Obviously, the information contained in the data cannot be expanded by using the same data more than once for a desired computation.

In actual practice, the averaging procedure is often accomplished by continuous smoothing of the squared amplitude signal with an RC low pass filter. For this case, from Ref. [1], Section 6.1.7, the equivalent averaging time T_1 is given by

$$T_1 = 2K \tag{13.20}$$

where K is the RC time constant of the low pass filter. Equation (13.20) assumes the signal has been applied for three or four time constants before a reading is taken, and that the time constant K is relatively long compared to the period of the lowest frequency of interest. Substituting Eq. (13.20) into Eq. (13.14), the following result is obtained for RC averaging type instruments.

$$\epsilon \approx \frac{1}{\sqrt{2BK}} \qquad K \leq \frac{T}{2} \tag{13.21}$$

In Eq. (13.21), ϵ defines the normalized standard error of the continuous estimate at any instant of time. Once again, ϵ can never be less than the value obtained by using the actual record length T in Eq. (13.14). Hence, if K is greater than T/2, the normalized standard error of the continuous estimate at any instant of time is fixed by the record length T and given by Eq. (13.14).

Additional information on the practical considerations of power spectra estimation is presented in Ref. [1], Section 7.4.

13.2 DESIGN OF EXPERIMENTS AND PROCEDURES

13.2.1 General Design and Procedures

The general purpose of these experiments is to verify the theoretical expression for the statistical uncertainty of power spectral density estimates, as given by Eq. (13.8). The basic procedure is to gather a series of N number of statistically independent power spectral density estimates, $\hat{G}$*, for a specific set of measurement parameters, B, f_c, and T_1. A variance for the series of estimates is computed and a value for ϵ^2 is determined. The procedure is repeated for different values of B, f_c, and T_1. The resulting set of empirical values for ϵ^2 are then tested for equivalence to the theoretical expression given by Eq. (13.8).

*The notation $\hat{G}$ means a power spectral density estimate at a specific center frequency, f_c; that is, $\hat{G} = \hat{G}(f)$ for $f = f_c$.

To be more specific, for each set of N number of statistically independent estimates $\hat{G}$, a sample mean and variance are computed as follows.

$$\bar{G} = \frac{1}{N} \sum_{i=1}^{N} \hat{G}_i \tag{13.22a}$$

$$s^2 = \frac{1}{N} \sum_{i=1}^{N} (\hat{G}_i - \bar{G})^2 = \frac{1}{N} \sum_{i=1}^{N} \hat{G}_i^2 - (\bar{G})^2 \; * \tag{13.22b}$$

The expected values for the above sample mean and variance are the power spectral density, G, and the variance of the estimate, $\sigma_{\hat{G}}^2$. That is,

$$E\left[\bar{G}\right] = G \tag{13.23a}$$

$$E\left[s^2\right] = \left(\frac{N-1}{N}\right) \sigma_{\hat{G}}^2 \approx \sigma_{\hat{G}}^2 \text{ for large N.} \tag{13.23b}$$

From Eq. (13.8), the uncertainty ϵ^2 is given by

$$\epsilon^2 \approx \frac{\sigma_{\hat{G}}^2}{G^2} = \frac{E\left[s^2\right]}{\left\{E\left[\bar{G}\right]\right\}^2} \tag{13.24}$$

Then, from the experimental data gathered, an estimate for the uncertainty ϵ^2 is given by

$$\hat{\epsilon}^2 = \frac{s^2}{(\bar{G})^2} \tag{13.25}$$

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*A biased expression for s^2 is employed here so that all statistical procedures to follow will be consistent with procedures outlined in Ref. [1].

13.2.2 Detailed Test Procedure

The random signal source used for these experiments has an approximately Gaussian probability density function and uniform power spectrum over the frequency range of interest. Power spectral density estimates are obtained using a commercial analyzer which integrates and averages instantaneous squared amplitudes (estimates mean square values) either by linear integration or by smoothing with a low pass RC filter. Power spectra estimates are gathered using both techniques for mean square level determination. However, emphasis is placed on estimates obtained using continuous RC averaging because this is the more common technique followed in actual practice. For this case, the continuously averaged estimate is recorded to obtain $\hat{G}$ versus time. Specific values of $\hat{G}$ are obtained by reading values from the continuous plot at equally spaced time intervals. The time interval between readings is selected to be at least 4K seconds to assure the $\hat{G}$ values are statistically independent. The details of all instruments, test set-up, and calibrations are presented in Section 13.3.

A total of $N = 61$ statistically independent estimates $\hat{G}$ are gathered for each of $M = 11$ different sets of values for B, f_c, and T_1, as follows:

A. Averaging accomplished by linear integration.
   $B = 56$ cps; $f_c = 1000$ cps; $T_1 = 1.0$ seconds.

B. Averaging accomplished by RC filtering.
   $B = 56$ cps; $f_c = 1000$ cps.

   B-1. $K = 0.84$ seconds ($T_1 = 1.7$ seconds)
   B-2. $K = 0.40$ seconds ($T_1 = 0.80$ seconds)
   B-3. $K = 0.13$ seconds ($T_1 = 0.26$ seconds)

C. Averaging accomplished by RC filtering.
   $B = 56$ cps; $K = 0.84$ seconds ($T_1 = 1.7$ seconds)

   C-1. $f_c = 100$ cps
   C-2. $f_c = 500$ cps
   C-3. $f_c = 5{,}000$ cps
   C-4. $f_c = 10{,}000$ cps

D. Averaging accomplished by RC filtering.

f_c = 1000 cps; K = 0.84 seconds (T_1 = 1.7 seconds).

D-1. B = 28 cps

D-2. B = 14 cps

D-3. B = 5.5 cps

Note that experiment B-1 provides an additional case for the experiments in both C and D.

The various values selected for B and T_1 are designed to cover the normal range of bandwidths and averaging times used for power spectral density analysis in actual practice. The various values for f_c are designed to cover the frequency range of predominate acoustic and vibration response in modern flight vehicles. The determination of the frequency bandwidths and averaging times employed is discussed in Section 13.4.

13.2.3 Statistical Hypothesis Tests

In order to confirm the theoretical definition for ϵ^2 given by Eq. (13.8), the eleven experimentally determined values, $\hat{\epsilon}^2$, are tested for equivalence to the theoretical value for ϵ^2. To perform a test for equivalence it is necessary to define a sampling distribution for the estimates, $\hat{\epsilon}^2$. From Eq. (13.25), it is seen that $\hat{\epsilon}^2$ is a function of a sample variance s^2 and a sample mean $\overline{G}$, which are in turn associated with experimentally determined values for a bandwidth B and an averaging time T_1. All of these quantities are in reality random variables.

The sample variance s^2 has a distribution associated with a chi-squared distribution as follows

$$\frac{s^2}{\sigma_{\hat{G}}^2} \sim \frac{\chi^2}{61} \tag{13.26}$$

where χ^2 is a chi-squared distribution with (N - 1) = 60 degrees of freedom.

The sample mean $\overline{G}$ is effectively a power spectral density estimate based upon $Nn = 61(2BT_1)$ number of events. Thus, $\overline{G}$ has a sampling distribution given by

$$\frac{\overline{G}}{G} \sim \frac{\chi^2}{(61n-1)} \tag{13.27}$$

where χ^2 has $(61n - 1)$ degrees of freedom. The parameters B and T_1, which affect s^2 and $\overline{G}$, also have some unknown variability associated with their determinations. The true sampling distribution for $\hat{\epsilon}^2$ in Eq.(13.25) is technically a function of all these variabilities.

In Eq. (13.25), the contribution of the term $(\overline{G})^2$ to the total variance of $\hat{\epsilon}^2$ is less than 20% of the contribution of the term s^2 to the total variance of $\hat{\epsilon}^2$ if $BT_1 > 10$, as is the case for all experiments performed herein. This relationship is developed in Section 13.5.3. Furthermore, the variances for the values for B and T_1 are believed to be negligible compared to the variance of s^2. Hence, it will be assumed for simplicity that s^2 is the only random variable in the experimental procedure for estimating ϵ^2 by Eq. (13.25). The effect of this assumption will be discussed later.

The distribution of s^2 is given by Eq. (13.26), which may be written using the relationships from Eq. (13.25) as follows.

$$\frac{\hat{\epsilon}^2}{\sigma_{\hat{G}}^2/(\overline{G})^2} \sim \frac{\chi^2}{61} \tag{13.28}$$

If $\overline{G}$ is considered the expected value G, from Eq. (13.8), the denominator of Eq. (13.28) is equal to the expected value for the uncertainty ϵ^2. Then,

$$\frac{\hat{\epsilon}^2}{\epsilon^2} \sim \frac{\chi^2}{61} \tag{13.29}$$

Let it be hypothesized that $\hat{\epsilon}^2$ and ϵ^2 are equivalent. That is,

$$H_0 : \hat{\epsilon}^2 = \epsilon^2 \tag{13.30}$$

Let the null hypothesis H_0 in Eq. (13.30) be tested at the $\alpha = 5\%$ level of significance. From Ref. [3], for $\alpha = 5\%$ and 60 degrees of freedom, $\chi^2_{\alpha/2} = 83.3$ and $\chi^2_{1-\alpha/2} = 40.5$. Then, the acceptance region for H_0 will be bounded by

$$\frac{40.5}{61} = 0.66 \leq \frac{\hat{\epsilon}^2}{\epsilon^2} \leq 1.37 = \frac{83.3}{61} \qquad (13.31)$$

If the ratio falls within the above limits, H_0 is accepted. If the ratio falls outside the above limits, H_0 is rejected and there is reason to suspect that $\hat{\epsilon}^2 \neq \epsilon^2$.

The Type I error (risk of rejecting H_0 when in fact it is true) is, of course, 5%. The Type II error (risk of accepting H_0 when in fact it is false) is a function of the level of significance α and the sample size N. The general procedures for computing the Type II error for hypothesis tests are developed in Ref. [1], Section 5.1. A detailed illustration of the procedures is given in Section 12.2.2. These experiments are designed such that the sample size of $N = 61$ tested at the $\alpha = 5\%$ level of significance gives a Type II error of 5% for detecting a 2:1 difference between $\hat{\epsilon}^2$ and ϵ^2 (1.4:1 difference between $\hat{\epsilon}$ and ϵ).

Consider now the assumption employed to arrive at Eq. (13.29); namely, that B, T_1, and $\overline{G}$ are exact values for the bandwidth, the averaging time, and the power spectral density being measured. The obvious effect of this assumption is to reduce the predicted variance of the random variable $\hat{\epsilon}^2$. In other words, the variability of $\hat{\epsilon}^2$ is actually greater than predicted by Eq. (13.29), and the region of acceptance for a test of H_0 at the 5% level of significance is actually greater than predicted by Eq. (13.31). The net result is that the Type I and Type II errors for the test of H_0 are both larger than the 5% value for which the experiments are designed. Thus, it can only be said that the Type I and Type II errors for the test of H_0 in Eq. (13.30) are <u>at least</u> 5%.

13.3 INSTRUMENTATION

13.3.1 Instruments and Test Set-Up

The laboratory instruments employed for these experiments are listed in Table 13.1. A block diagram for the test set-up is illustrated in Figure 13.2. Except for the random noise generator, all instruments are the property of the Norair Division of Northrop Corporation, and were in current calibration at the time of the experiments.

The random noise generator (Item A) is used as the source of a signal which is considered in these experiments to be a stationary random signal. The instrument generates noise with an approximately uniform power spectral density function over a frequency range from less than 100 cps to over 10000 cps (the frequency range of interest in these experiments). See Section 14.3 for details.

The sine wave generator (Item B) is used for calibration purposes as discussed in Sections 13.3.2 and 13.3.3. The frequency counter (Item C) is used to determine the frequency of the sine wave calibration signals with an accuracy of $\pm$ 1 cps.

The voltmeter (Item D) is used to measure the output rms voltage level of the random noise generator, or the sine wave generator, for reference purposes.

The power spectral density analyzer (Item E) is used to obtain power spectral density estimates at various center frequencies and with various bandwidths and averaging times, as discussed in Sections 13.3.2 and 13.3.3. The strip chart recorder (Item F) is used to record power spectral density time history samples for the signal to be investigated.

Item	Description	Manufacturer	Model No.	Serial No. *
A	Random Noise Generator	General Radio Co.	1390A	937
B	Audio Oscillator	Hewlett-Packard	202D	67117
C	Universal EPUT and Timer	Berkeley Division	7350	79528
D	True RMS Voltmeter	B&K Instruments	2409R	PR-13147-1
E	Power Spectral Density Analyzer	Technical Products Company	TP-626 TP-627 TP-633	NC44065
F	Strip Chart Recorder	Brush Instruments	Mark II	270822

*Except for Item A, the serial numbers refer to Northrop Corporation identification tags.

Table 13.1 Instruments Employed for the Experiments

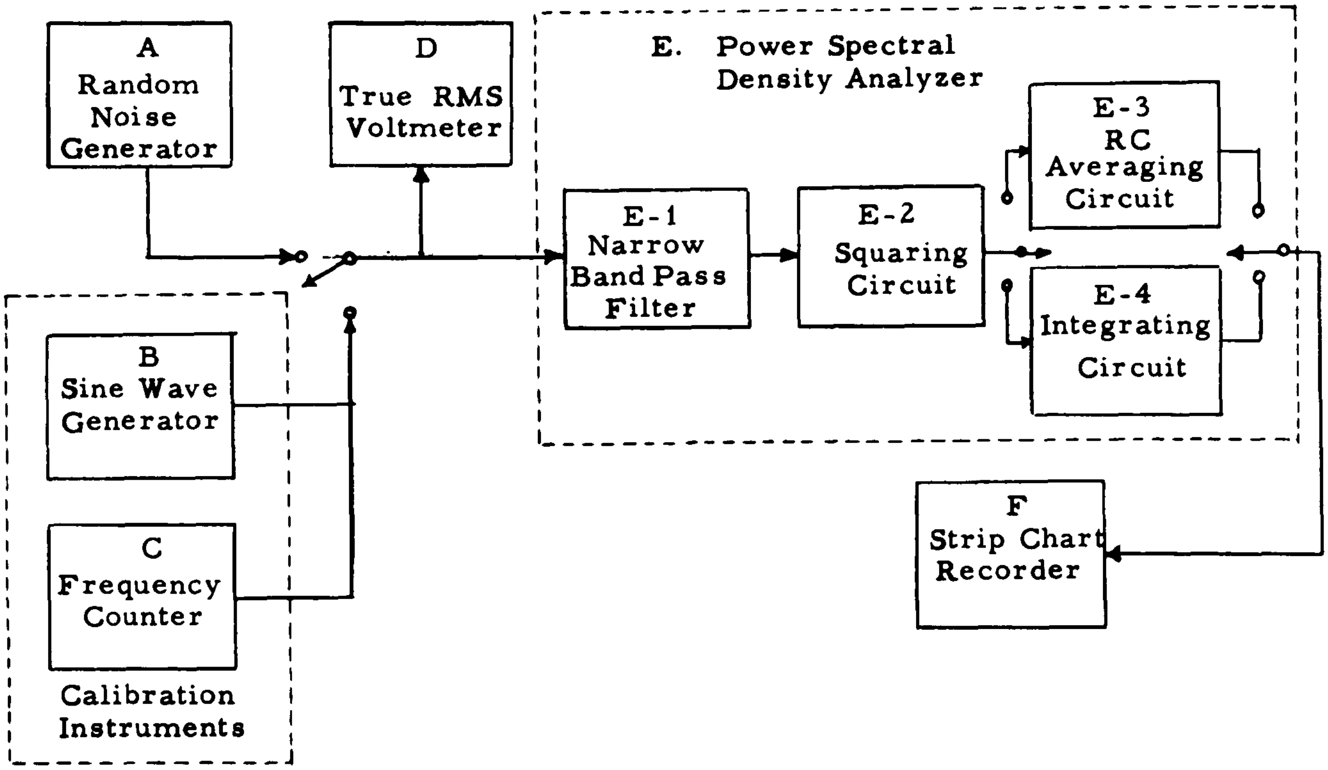

Figure 13.2 Block Diagram of Test Set-Up

13.3.2 Characteristics of Narrow Bandpass Filter

The power spectral density analyzer (Item E) is a heterodyne type instrument. Narrow band filtering is accomplished by transposing the input signal in frequency past a high frequency crystal filter. Several different bandwidths are available. For these experiments, four different filters are used.

The cutoff characteristics of each of the four filter selections have been checked by applying a sinusoidal signal from Item B with a known frequency read from Item C. The cutoff for each filter has been found to be very sharp, over 60 db per octave for a center frequency of 100 cps. The cutoff for each filter is a function of the deviation in cps from the center frequency, and is independent of the value of the center frequency. Thus, the cutoff rate in terms of db per octave increases as the center frequency increases. The selectivity of the filters over the center frequencies of interest is sufficiently sharp to assume the effective noise bandwidth of each filter is equal to the bandwidth between the half power points. The half power point bandwidths have been determined to be 56 cps, 28 cps, 14 cps, and 5.5 cps.

13.3.3 Determination of Averaging Time Constants

The power spectral density analyzer (Item E) is equipped with a continuously variable averaging time constant from 0.1 seconds to 100 seconds. Only three different averaging times are used in these experiments; namely, a dial indicated time constant of one second, 0.5 seconds, and 0.1 seconds. The actual value of the time constant K associated with each of the three dial settings is established as follows.

Referring to Figure 13.2, a sinusoidal voltage signal from Item B is applied to the test set-up and read out as a power spectral density measurement at Item F. Since the applied signal is periodic, there is no statistical uncertainty in the resulting power spectral density indications. Item F is operated with a high chart speed (125 mm/sec) so that the response time characteristics of the test set-up are accurately defined. The time constant K is determined by measuring the time required for a power spectral density indication to rise from zero to 63.4% of the final steady state value when the sinusoidal signal is applied instantaneously,

or the time required for the power spectral density indication to fall 63.4% from the steady state value when the sinusoidal signal is removed instantaneously. This is done at several different frequencies to obtain at least 10 measurements of K for each dial setting of interest on Item E.

Using the above procedure, the time constants associated with the three dial settings of interest have been determined to be as follows.

1.0 second setting - K = 0.84 seconds
0.5 second setting - K = 0.40 seconds
0.1 second setting - K = 0.13 seconds

The standard deviation of the measurements is such that only two significant figures are considered justified.

It should be mentioned that the accuracy of the values determined for K is a function of the accuracy of the paper speed control of the recorder, Item F. The accuracy of the recorder paper speed has been checked by directly recording sine waves of various frequencies determined to $\pm$ 1 cps using Item C. The results indicate the error in the indicated paper speed to be insignificant.

13.4 RESULTS OF EXPERIMENTS

13.4.1 Power Spectral Density Data

The 61 power spectral density estimates gathered for each of the 11 experiments are presented in Tables 13.2, 13.3, and 13.4. The test designations refer to test procedures defined in Section 13.2. A picture of some typical test data for power spectral density estimates obtained by continuous RC averaging is shown in Figure 13.3.

13.4.2 Sampling Statistics for Power Spectral Density Estimates

The sample mean and variance for the 61 power spectral density estimates gathered for each of the 11 experiments are presented in Table 13.5. The sample mean and variance are computed using Eq. (13.22). It should be noted that no effort was made to use the same power spectral density for the different experiments, which is why the values for $\overline{G}$ vary.

Power Spectral Density Estimates $\hat{G}$ for Record Lengths of T Seconds or Time Constants of K Seconds

$B = 56$ cps ; $f_c = 1000$ cps

Estimate Number	A T = 1.0	B-1 K = 0.84	B-2 K = 0.40	B-3 K = 0.13	Estimate Number	A T = 1.0	B-1 K = 0.84	B-2 K = 0.40	B-3 K = 0.13
1	0.85	5.10	4.60	4.75	31	0.90	5.24	5.02	3.50
2	0.95	4.20	4.74	2.71	32	0.80	4.57	4.36	4.58
3	0.78	5.13	4.28	4.06	33	0.90	4.10	5.79	4.25
4	0.85	4.33	4.19	3.50	34	0.97	5.25	5.36	4.27
5	0.84	4.79	5.00	4.00	35	0.88	4.58	5.39	3.00
6	0.78	4.58	5.95	3.71	36	0.84	5.05	6.88	5.95
7	0.77	4.83	4.02	4.41	37	0.85	5.28	4.92	1.92
8	0.93	4.44	4.40	3.16	38	0.66	4.60	4.65	5.31
9	1.01	4.00	4.72	5.00	39	0.98	5.53	6.22	1.71
10	1.08	5.01	4.30	2.75	40	1.19	4.04	4.20	3.70
11	0.93	3.78	4.24	5.60	41	0.73	4.52	4.62	3.40
12	0.87	3.95	5.00	5.39	42	1.10	4.88	4.34	4.50
13	0.87	4.51	4.55	3.46	43	0.96	4.83	5.32	2.01
14	0.91	5.03	4.97	5.70	44	0.74	4.70	4.24	4.38
15	0.85	4.52	5.09	3.10	45	1.05	4.61	4.43	4.39
16	0.69	5.64	5.40	2.92	46	0.75	4.42	4.48	3.59
17	0.84	4.30	4.75	3.45	47	0.98	3.95	5.27	3.72
18	0.78	3.94	5.12	4.71	48	0.89	5.16	5.58	3.71
19	0.85	4.46	4.99	2.98	49	0.83	5.43	4.99	4.32
20	1.06	4.90	3.97	3.15	50	0.83	4.78	5.27	4.53
21	0.85	4.65	4.78	5.95	51	0.84	4.40	5.76	4.01
22	0.89	4.42	6.12	2.54	52	0.72	4.52	3.65	3.22
23	0.95	4.96	5.62	2.69	53	0.87	4.45	4.61	1.80
24	0.95	5.07	5.23	3.61	54	0.75	4.08	4.42	3.50
25	1.01	4.15	6.35	4.52	55	1.00	5.27	3.80	2.30
26	0.97	4.20	5.28	2.90	56	0.97	4.40	4.02	3.60
27	0.85	4.62	6.12	2.26	57	0.86	4.08	4.21	2.44
28	0.94	4.54	4.30	3.97	58	0.74	4.20	5.30	4.78
29	0.94	4.45	5.60	3.44	59	0.89	3.96	5.56	3.80
30	0.93	4.28	5.48	5.12	60	0.69	4.24	4.70	4.00
					61	0.71	4.50	5.29	4.07

Table 13.2 Power Spectral Density Estimates for Different Averaging Times

Power Spectral Density Estimates $\hat{G}$ for Filter Center Frequencies of f_c cps

B = 56 cps ; K = 0.84 seconds

Estimate Number	C-1 f_c=100	C-2 f_c=500	C-3 f_c=5000	C-4 f_c=10000	Estimate Number	C-1 f_c=100	C-2 f_c=500	C-3 f_c=5000	C-4 f_c=10000
1	4.55	3.68	4.79	5.53	31	4.08	3.73	4.00	5.46
2	5.34	3.67	4.45	4.80	32	4.93	4.02	5.29	6.19
3	5.13	4.11	4.78	5.08	33	4.90	4.51	4.83	5.35
4	4.29	3.83	5.76	4.73	34	4.75	3.66	4.99	5.58
5	4.93	3.50	4.68	4.41	35	4.10	4.08	4.65	4.60
6	4.52	3.69	3.88	5.22	36	5.02	4.32	5.25	5.21
7	5.04	3.98	5.47	5.13	37	4.25	4.20	5.52	4.77
8	4.36	3.82	4.59	5.12	38	4.66	3.70	4.20	4.59
9	4.40	3.83	4.90	4.84	39	4.23	3.89	5.25	4.61
10	5.30	3.51	5.79	4.51	40	4.12	3.52	4.48	4.50
11	4.43	4.28	4.87	4.72	41	4.75	3.69	4.80	5.05
12	4.79	3.51	5.18	4.95	42	4.11	4.20	5.31	5.02
13	3.92	3.19	3.81	4.60	43	4.30	4.12	5.60	4.11
14	4.47	3.81	4.40	4.84	44	5.70	4.23	4.35	4.55
15	4.12	4.06	5.25	3.98	45	4.13	4.55	4.70	4.35
16	5.15	3.67	4.58	4.92	46	4.68	4.55	3.79	4.68
17	5.03	4.10	4.83	5.79	47	5.16	4.00	4.30	5.01
18	4.84	3.88	5.00	4.79	48	4.45	4.00	4.60	4.90
19	4.60	4.27	4.87	4.60	49	4.60	3.44	4.38	4.55
20	4.28	3.75	4.55	4.99	50	3.74	3.75	5.23	4.49
21	3.88	4.00	4.81	3.83	51	4.92	4.16	4.60	5.30
22	4.71	3.81	4.20	6.22	52	4.73	4.01	5.35	4.08
23	4.19	4.01	3.41	4.40	53	4.52	3.93	4.31	3.78
24	4.80	3.50	5.02	4.92	54	4.45	4.69	4.43	5.42
25	3.92	3.71	5.21	5.03	55	4.89	4.00	5.09	4.00
26	5.25	4.00	4.82	4.65	56	5.22	4.12	4.20	4.29
27	4.08	4.83	3.74	4.30	57	5.11	4.00	4.48	4.40
28	4.83	3.13	4.60	5.16	58	4.70	3.98	4.80	4.32
29	4.81	4.41	4.88	4.80	59	4.75	4.21	5.21	3.77
30	4.82	3.48	5.80	4.78	60	4.38	3.70	5.02	4.60
					61	5.20	3.79	5.03	4.81

Table 13.3 Power Spectral Density Estimates for Different Center Frequencies

Power Spectral Density Estimate $\hat{G}$ for Filter Bandwidths of B cps

f_c = 1000 cps ; K = 0.84 seconds

Estimate Number	D-1 B=28	D-2 B=14	D-3 B=5.5	Estimate Number	D-1 B=28	D-2 B=14	D-3 B=5.5
1	4.80	3.20	4.90	31	3.23	4.85	7.50
2	2.85	4.42	3.98	32	3.60	5.53	3.43
3	3.13	4.98	4.63	33	2.89	4.60	4.22
4	3.96	4.40	4.32	34	3.42	4.41	3.53
5	4.02	4.61	2.55	35	3.54	3.95	6.80
6	3.69	4.69	2.24	36	4.50	3.53	3.44
7	4.16	5.10	5.68	37	3.50	4.08	4.40
8	3.28	4.62	3.16	38	4.40	3.22	3.32
9	3.83	4.16	3.39	39	3.52	4.80	5.30
10	3.34	5.00	4.27	40	3.48	4.19	4.98
11	4.56	3.45	3.68	41	5.39	5.76	5.26
12	3.13	5.10	3.72	42	3.70	3.42	4.68
13	3.70	3.99	4.21	43	3.01	4.55	6.50
14	4.67	6.08	2.35	44	4.08	4.90	7.42
15	4.20	5.81	6.82	45	3.61	5.50	7.78
16	3.79	4.82	5.97	46	3.75	4.30	2.27
17	4.14	4.70	7.02	47	4.34	3.59	4.21
18	4.22	5.02	4.39	48	4.52	4.63	4.08
19	3.99	4.60	6.31	49	3.13	5.48	2.93
20	3.90	4.93	2.82	50	4.02	3.97	3.73
21	3.78	5.88	3.15	51	4.18	3.41	3.60
22	4.32	4.67	4.11	52	4.80	6.55	5.00
23	3.60	6.40	5.80	53	3.75	5.90	6.90
24	5.01	4.32	3.92	54	4.89	5.68	5.70
25	4.81	4.06	6.95	55	4.00	4.94	4.15
26	3.32	6.02	5.12	56	4.32	4.52	4.06
27	4.93	5.30	7.42	57	3.10	4.75	2.75
28	3.42	5.62	2.73	58	4.12	5.03	2.85
29	3.17	4.10	3.89	59	4.48	5.40	3.34
30	3.15	3.01	5.36	60	3.46	4.92	4.26
				61	4.19	3.61	2.80

Table 13.4 Power Spectral Density Estimates for Different Filter Bandwidths

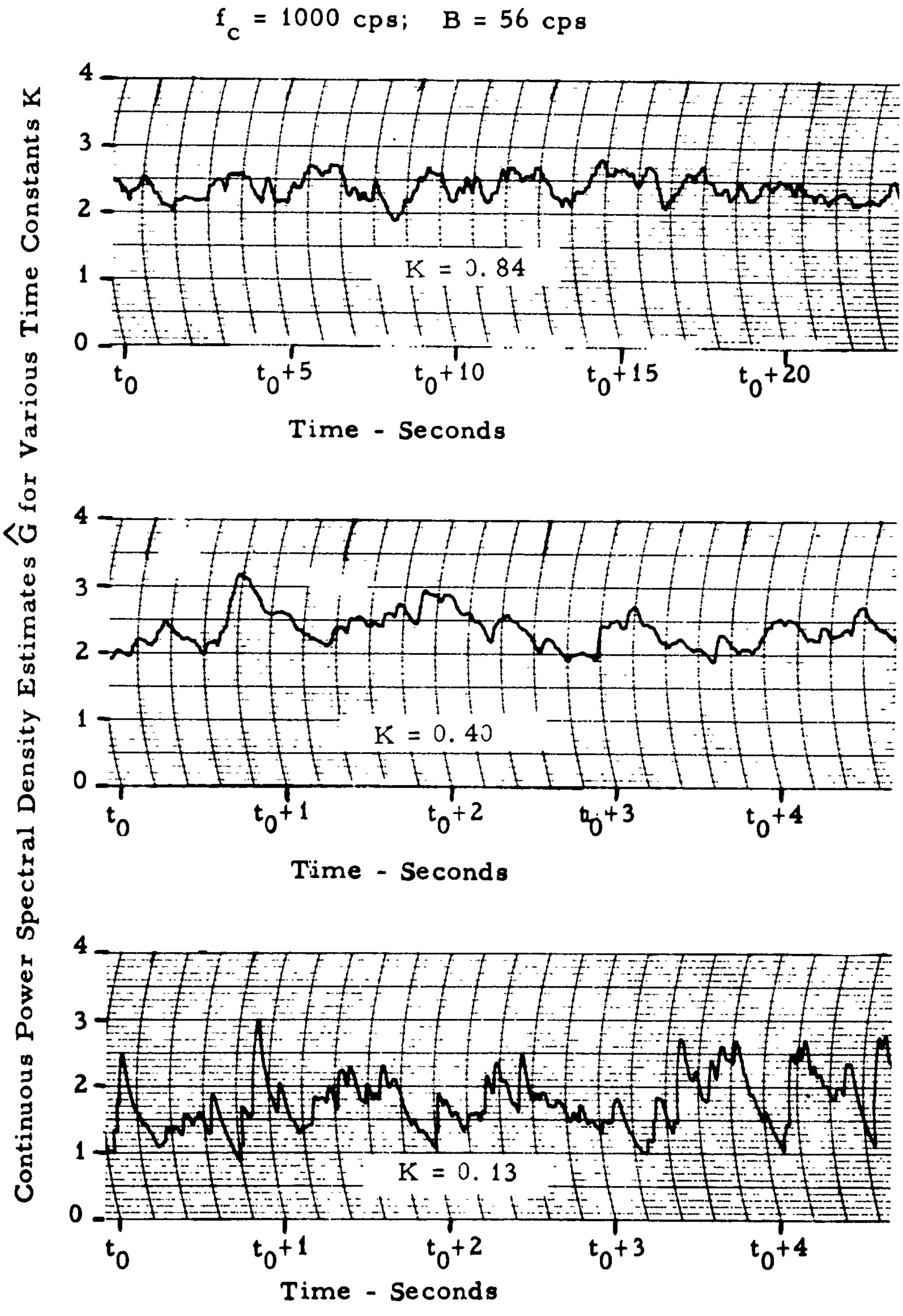

Figure 13.3 Examples of Actual Power Spectral Density Data

Sample Mean $\bar{G}$ and Sample Variance s^2 For Power Spectral Density Estimates $\hat{G}$			
Description of Experiments (Test Procedures)	Sample Size N	Sample Mean $\bar{G}$	Sample Variance s^2
Procedure A (Table 2)	61	0.879	0.0117
Procedures B (Table 2	↓		
B-1		4.60	0.191
B-2		4.95	0.460
B-3		3.77	1.034
Procedures C (Table 3)			
C-1		4.63	0.178
C-2		3.93	0.114
C-3		4.77	0.267
C-4		4.79	0.262
Procedures D (Table 4)			
D-1		3.90	0.346
D-2		4.71	0.665
D-3		4.52	2.19

Table 13.5 Sample Means and Variances for Power Spectral Density Estimates

Statistical Test of the Null Hypothesis $H_0 : \hat{\epsilon}^2 = \epsilon^2$				
Experiment (Test Procedure)	$\epsilon^2 = \frac{1}{BT}$ $\epsilon^2 = \frac{1}{2BK}$	$\hat{\epsilon}^2 = \frac{s^2}{\bar{G}^2}$	$\frac{\hat{\epsilon}^2}{\epsilon^2}$	Region of Acceptance for H_0 : $0.66 \leq \frac{\hat{\epsilon}^2}{\epsilon^2} \leq 1.37$
A	0.018	0.015	0.83	accepted
B-1	0.011	0.0090	0.82	accepted
B-2	0.022	0.019	0.86	accepted
B-3	0.068	0.073	1.1	accepted
C-1	0.011	0.0083	0.75	accepted
C-2	0.011	0.0074	0.67	accepted
C-3	0.011	0.012	1.1	accepted
C-4	0.011	0.011	1.0	accepted
D-1	0.023	0.023	1.0	accepted
D-2	0.043	0.030	0.70	accepted
D-3	0.094	0.107	1.1	accepted

Table 13.6 Statistical Tests for Equivalence

13.4.3 Results of Statistical Tests

The tests of the null hypothesis, $H_0 : \hat{\epsilon}^2 = \epsilon^2$ are presented in Table 13.6. Only two significant figures are carried for the tests because the accuracy of the measurements of B and T_1 do not justify more than two significant figures in the values for ϵ^2.

The null hypothesis H_0 is accepted for all 11 experiments with a minimum Type I and Type II error of 5%. Thus, there is no reason to question the validity of Eq. (13.8), $\epsilon^2 = 1/BT$, for the range of parameter values tested.

13.5 DISCUSSION OF RESULTS

13.5.1 Variability of Estimates for ϵ^2

In Section 13.2.3, the experimentally determined values for $\hat{\epsilon}^2$ are considered to have a distribution associated with χ^2 as shown in Eq. (13.29). The assumption employed to arrive at this conclusion is that all parameters used to determine the ratio $\hat{\epsilon}^2/\epsilon^2$ are exact values except for s^2. Then the variability of the ratio should actually be greater than predicted by a χ^2 distribution. It is interesting to see if this effect is apparent in the results presented in Section 13.4.3.

First, consider the variance of the ratio $\hat{\epsilon}^2/\epsilon^2$ predicted by the χ^2 distribution. From Eq. (13.29), the variance of the ratio is given by

$$\text{Var}\left(\frac{\hat{\epsilon}^2}{\epsilon^2}\right) = \text{Var}\left(\frac{\chi^2}{61}\right) = \frac{\text{Var}(\chi^2)}{(61)^2}$$

where χ^2 has 60 degrees of freedom. The variance of a χ^2 distribution is two times the number of degrees of freedom, or 120 for χ^2 with 60 degrees of freedom.

Then,

$$\mathrm{Var}\ \frac{\hat{\epsilon}^2}{\epsilon} = \frac{120}{3721} = 0.0322 \tag{13.32}$$

Now consider the actual variance of the ratio $\hat{\epsilon}^2/\epsilon^2$ obtained in the experimental results. From the 11 values in Table 13.6 and Eq. (13.22),

$$\begin{aligned} \mathrm{Var}(r) &= \frac{1}{11}\sum_{i=1}^{11} (r_i - \bar{r})^2 = 0.03 \\ r &= \frac{\hat{\epsilon}^2}{\epsilon^2} \end{aligned} \tag{13.33}$$

Comparing the results of Eq. (13.33) with Eq. (13.32), it appears the assumption employed to arrive at Eq. (13.29) is completely reasonable.

13.5.2 Uncertainty of Mean Square Estimates

The mean square value σ_x^2 for a random signal between the frequency limits f_1 and f_2 is given by Eq. (13.5) as

$$\sigma_x^2(f_1, f_2) = \int_{f_1}^{f_2} G(f)\, df \tag{13.5'}$$

When a mean square value is estimated from a measured power spectrum, the estimate $\overline{x^2}$ is given by

$$\overline{x^2}(f_1, f_2) = \int_{f_1}^{f_2} \hat{G}(f)\, df = \sum_{i=1}^{N} \hat{G}_i B_i \tag{13.34}$$

and

$$E\left[\overline{x^2}(f_1, f_2)\right] = \sigma_x^2(f_1, f_2) = \sum_{i=1}^{N} G_i B_i \tag{13.35}$$

Here, N is the number of center frequencies required to cover the frequency range $(f_2 - f_1)$, $\hat{G}_i$ is the power spectral density measured at the ith center frequency, and B_i is the bandwidth associated with the ith center frequency.

Now consider the uncertainty of the mean square estimate $\overline{x^2}$. The variance of a sum of random variables is the sum of the variances. That is, if

$$x = y_1 + y_2 + \dots + y_n$$

then $\text{Var}(x) = \text{Var}(y_1) + \text{Var}(y_2) + \dots + \text{Var}(y_n)$. Then the variance of the mean square estimate in Eq. (13.34) will be

$$\text{Var}\left[\overline{x^2}(f_1, f_2)\right] = \sum_{i=1}^{N} \text{Var}(\hat{G}_i, B_i) = \sum_{i=1}^{N} B_i^2 \ \text{Var}(\hat{G}_i) \qquad (13.36)$$

where, from Eq. (13.8)

$$\text{Var}(\hat{G}_i) = \sigma^2_{\hat{G}_i} = \frac{G_i^2}{B_i T} \qquad (13.37)$$

Thus

$$\text{Var}\left[\overline{x^2}(f_1, f_2)\right] = \frac{1}{T} \sum_{i=1}^{N} G_i^2 B_i \qquad (13.38)$$

From Eqs. (13.35) and (13.38), the normalized variance of the mean square estimate is now given by

$$\epsilon^2 = \frac{\text{Var}\left[\overline{x^2}(f_1, f_2)\right]}{\left[\sigma_x^2(f_1, f_2)\right]^2} = \frac{\sum_{i=1}^{N} G_i^2 B_i}{T\left(\sum_{i=1}^{N} G_i B_i\right)^2} \qquad (13.39)$$

Equation (13.39) is a general result which applies to arbitrary power spectral density G_i and arbitrary bandwidths B_i.

For the simple case where a uniform power spectral density $G_i = G$ is measured between f_1 and f_2 using a constant bandwidth filter $B_i = B$, Eq. (13.39) becomes

$$\epsilon^2 = \frac{NG^2B}{T(NGB)^2} = \frac{1}{TNB} = \frac{1}{(f_2 - f_1)T} \qquad (13.40)$$

since for this case $NB = (f_2 - f_1)$.

The results of Eq. (13. 40) illustrate that the normalized variance for a mean square estimate is exactly the same as for a power spectral density estimate (as long as the power spectrum is uniform over the frequency range being considered). This result should not be surprising since power spectral density measurements are effectively mean square level estimates in narrow bandwidths.

The same result can be arrived at more directly by considering the case where $B = (f_2 - f_1)$. From Eqs. (13. 6) and (13. 8),

$$\overline{x^2}(B) = B\hat{G}$$

$$\mathrm{Var}\left[\overline{x^2}(B)\right] = B^2 \ \mathrm{Var}\ \hat{G} = \frac{B^2 G^2}{BT}$$

From Eq. (13. 5),

$$E\left[\overline{x^2}(B)\right] = \sigma_x^2(B) = BG$$

$$\epsilon^2 = \frac{\mathrm{Var}\left[\overline{x^2}(B)\right]}{\left[\sigma_x^2(B)\right]^2} = \frac{1}{BT}$$

13. 5. 3 Details of Experimental Error Analysis

Any analytic function of n variables, $f(y_1, y_2, \ldots, y_n)$, can be expanded into a Taylor series about a particular point, say $y_1^*, y_2^*, \ldots, y_n^*$, as follows.

$$f(y_1, y_2, \ldots, y_n) = f(y_1^*, y_2^*, \ldots, y_n^*) + (y_1 - y_1^*) \left.\frac{\partial f}{\partial y_1}\right|_{y_1^*, y_2^*, \ldots, y_n^*} + \ldots$$
$$+ (y_n - y_n^*) \left.\frac{\partial f}{\partial y_n}\right|_{y_1^*, y_2^*, \ldots, y_n^*} + \text{higher order terms.} \tag{13. 41}$$

Neglecting higher order terms and assuming the random variables are independent with expected values at $y_1^*, y_2^*, \ldots, y_n^*$, respectively, the variance of the function is approximated by

$$\mathrm{Var}\left[f(y_1, y_2, \ldots, y_n)\right] \approx \mathrm{Var}(y_1)\left[\left.\frac{\partial f}{\partial y_1}\right|_{y_1^*, y_2^*, \ldots, y_n^*}\right]^2 + \ldots$$
$$+ \mathrm{Var}(y_n)\left[\left.\frac{\partial f}{\partial y_n}\right|_{y_1^*, y_2^*, \ldots, y_n^*}\right]^2 \tag{13. 42}$$

Now consider the function $\hat{\epsilon}^2$ as defined in Eq. (13. 25). The term $\hat{\epsilon}^2$ is a function of two random variables, $\bar{G}$ and s^2. From Eq. (13. 42),

$$\mathrm{Var}\,(\hat{\epsilon}^2) = \mathrm{Var}\left[\frac{s^2}{(\bar{G})^2}\right] = \mathrm{Var}\,(s^2)\left[\frac{1}{(\bar{G}^*)^2}\right]^2 + \mathrm{Var}\,(\bar{G})\left[\frac{-2(s^2)^*(\bar{G}^*)}{(\bar{G}^*)^4}\right]^2 \tag{13.43}$$

From Eqs. (13. 12), (13. 13), (13. 26), and (12. 27), and noting that the variance of χ^2 is equal to twice the number of degrees of freedom, the following relationships are true

$$(s^2)^* = \left(\frac{N-1}{N}\right)\sigma_{\hat{G}}^2 = \frac{2(N-1)G^2}{N(n-1)}$$

$$(\bar{G})^* = G$$

$$\mathrm{Var}\,(s^2) = \mathrm{Var}\left(\frac{\sigma_{\hat{G}}^2\,\chi^2}{N}\right) = \frac{2(N-1)\sigma_{\hat{G}}^4}{N^2}$$

$$\mathrm{Var}\,(\bar{G}) = \mathrm{Var}\left(\frac{G\chi^2}{Nn-1}\right) = \frac{2G^2}{(Nn-1)}$$

For the case where both N and n are much greater than unity, the above relationships reduce to the following

$$\begin{aligned}
(s^2)^* &\approx \frac{2G^2}{n} \quad \text{and} \quad \sigma_{\hat{G}}^2 \approx \frac{2G^2}{n} \\
(\bar{G})^* &= G \\
\mathrm{Var}\,(s^2) &\approx \frac{2\sigma_{\hat{G}}^4}{N} \approx \frac{8G^4}{Nn^2} \\
\mathrm{Var}(\bar{G}) &= \frac{2G^2}{Nn}
\end{aligned} \tag{13.44}$$

Substituting the relationships from Eq. (13.44) into Eq. (13.43), the following results are obtained.

$$\mathrm{Var}(\hat{\epsilon}^2) = \frac{8G^4}{Nn^2}\left[\frac{1}{G^2}\right]^2 + \frac{2G^2}{Nn}\left[\frac{-4G^3}{nG^4}\right]^2 = \frac{8}{Nn^2}\left(1 + \frac{4}{n}\right) \tag{13.45}$$

From Eq. (13.45), the term (4/n) constitutes that portion of the variance for $\hat{\epsilon}^2$ contributed by $\bar{G}$. Clearly, as $n = 2BT_1$ becomes larger than 4, the effect of $\bar{G}$ on the variance of $\hat{\epsilon}^2$ diminishes. For example, if $n = 2BT_1 > 20$, the variance of $\hat{\epsilon}^2$ would be 20% greater than the variance computed assuming s^2 to be the only random variable. If $n > 40$, the variance of $\hat{\epsilon}^2$ would be only 10% greater, and so forth.

13.5.4 Normality Assumption for Power Spectra Estimates

Throughout this section, power spectral density estimates are assumed to have a distribution associated with χ^2. However, as the equivalent number of events n becomes large, say greater than 60, the χ^2 distribution is closely approximated by a normal distribution. Then for large n, it may be assumed for practical applications that power spectral density (or mean square) estimates are normally distributed with a normalized variance of ϵ given in Eq. (13.8). It is important to note that even if the vibration signal $x(t)$ does not have an approximately Gaussian probability density function, the normality assumption for power spectra estimates is still valid if n is large.

13.6 CONCLUSIONS

The theoretical expression for the uncertainty of power spectra estimates, as given by Eq. (13.8), has been investigated by carefully designed laboratory experiments. The practical validity of Eq. (13.8) was tested for analyzer bandwidths from 5.5 to 56 cps, center frequencies from 100 to 10000 cps, and averaging times from 0.26 to 1.68 seconds. Averaging by true integration and by smoothing with a low pass RC filter were both employed. All results support the practical validity of Eq. (13.8).

13.7 REFERENCES

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14. UNCERTAINTY OF PROBABILITY DENSITY ESTIMATES

14.1 THEORY OF PROBABILITY DENSITY ESTIMATION

14.1.1 Review of Mathematical Relationships

Consider a stationary random signal, $x(t)$. The probability that $x(t)$ assumes particular amplitude values between x and $x + \Delta x$ during a time interval of T seconds may be estimated by

$$\hat{P}(x, x+\Delta x) = \frac{1}{T}\sum_{j=1}^{k} \tau_i = \frac{\Delta\tau}{T} \qquad (14.1)$$

where τ_i is the time spent by the signal in the range $(x, x+\Delta x)$ during the ith entry to the range, as shown in Figure 14.1. The term $\Delta\tau/T$ is the total fractional portion of the time spent by the signal in the range $(x, x+\Delta x)$. It should be noted that $\Delta\tau$ will usually be a function of the amplitude x.

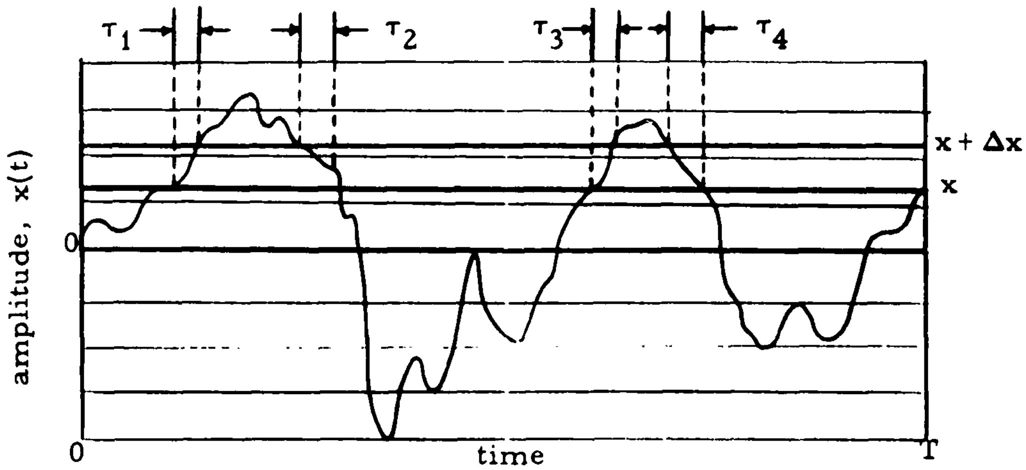

Figure 14.1 Sample Vibration Amplitude Time History Record

From Eq. (14.1), the estimated probability $\hat{P}$ will approach an exact probability P as the sample record length T approaches infinity. That is,

$$P(x, x+\Delta x) = \lim_{T\to\infty} \frac{\Delta\tau}{T} \qquad (14.2)$$

The probability density function is given by

$$p(x) = \lim_{\Delta x \to 0} \frac{P(x, x+\Delta x)}{\Delta x} = \lim_{\Delta x \to 0} \lim_{T \to \infty} \frac{1}{T} \left(\frac{\Delta \tau}{\Delta x} \right) \tag{14.3}$$

The probability density function p(x) then defines the probability P of amplitudes occurring between any two amplitude limits, x_1 and x_2, as follows.

$$P(x_1 < x < x_2) = \int_{x_1}^{x_2} p(x)\, dx \tag{14.4}$$

From Eq. (14. 3), the procedure of taking the limit as Δx approaches zero is beyond the capabilities of physical instruments and, of course, the record length T must always be finite. However, if Δx is small, an estimate for the probability density at any amplitude x for a sample record length T is given by

$$\hat{p}(x) \approx \frac{1}{T} \left(\frac{\Delta \tau}{\Delta x} \right) \quad \text{for small } \Delta x \tag{14.5}$$

Equation (14. 5) defines the basic operations which an analog instrument must accomplish to estimate the probability density function of a random signal from a sample record of length T.

1. Measurement of the total time, $\Delta \tau$, that the signal amplitude falls within a narrow amplitude window, Δx.
2. Division by the amplitude window, Δx.
3. Division by the total observation time (record length), T.

Of course, the center amplitude of the window, Δx, would have to be variable to cover the entire range of amplitudes under consideration.

14. 1. 2 Theoretical Evaluation of Estimation Uncertainty

The statistical uncertainty associated with amplitude probability density estimates has been investigated and defined by two different procedures in Sections 4. 9. 2 and 7. 5. 3 of Ref. [1]. Those discussions, however, only consider signals with a frequency range from DC to some high frequency cutoff, f_b. Furthermore, they employ simplifying

assumptions which produce results known to be very conservative. The uncertainty of probability density estimates is now considered in more detail for the general case where the signal is band-limited between any two frequencies, f_a and f_b.

The uncertainty in a probability density estimate at any amplitude x may be expressed by the variance of the estimate given by

$$\sigma^2_{\hat{p}(x)} = \frac{\sigma^2_p}{n} \tag{14.6}$$

where σ^2_p is the population variance and n is the equivalent number of events which the estimate is based upon. The definition of these two parameters will now be discussed.

(a) Number of Events - Method 1:

From classical sampling theory, Ref. [4], the number of events represented by a continuous random signal is given by $n = 2BT$ where B is an equivalent ideal bandwidth in cps and T is the available record length in seconds. To be more exact, T represents the total time the signal is actually observed and analyzed. For the problem at hand, T is only that time spent by the signal within the amplitude window Δx, since the signal is not actually observed and analyzed when the amplitudes are outside the window Δx. This actual analysis time is given by $\Delta \tau$ in Eq. (14.5). Thus, for analyzing a sample record of length T with an amplitude window of width Δx, the equivalent number of events becomes

$$n = 2B\,\Delta\tau \tag{14.7}$$

From Eq. (14.5), $\Delta\tau = \Delta x\,\hat{p}(x)\,T$. Substituting $\Delta\tau$ into Eq. (14.7) gives the following result.

$$n = 2\,\Delta x\,\hat{p}(x)\,BT \tag{14.8}$$

(b) Number of Events - Method 2:

Another expression for the equivalent of number of events n can be arrived at by a totally different approach as follows. For analyzing a sample record of length T with an amplitude window of width Δx, the total number of times that data is observed is equal to $\overline{\nu}_{(x,\,x+\Delta x)}\,T$ where

$\overline{\nu}_{(x, x+\Delta x)}$ is the number of crossings per second of the amplitude interval, $(x, x+\Delta x)$. The number of events may be thought of as the number of crossings of the interval $(x, x+\Delta x)$ multiplied by the width Δx of the interval. If the interval width Δx is small, the number of crossings of the interval $(x, x+\Delta x)$ is approximately equal to the number of crossings of the level x, denoted by $\overline{\nu}_x T$. Thus,

$$n = \Delta x \, \overline{\nu}_x \, T \qquad (14.9)$$

To permit further simplification of Eq. (14.9), assume the signal being investigated has an approximately Gaussian amplitude probability density function with a mean value of zero. For this case, from Ref. [2],

$$\overline{\nu}_x = \sqrt{2\pi} \; p(x) \, \overline{\nu}_0 \qquad (14.10)$$

Substituting Eq. (14.10) into Eq. (14.9) and noting that a measured probability density $\hat{p}(x)$ must be used, the following result is obtained.

$$n = 2.51 \, \Delta x \, \hat{p}(x) \, \overline{\nu}_0 \, T \qquad (14.11)$$

Comparing Eq. (4.11) and Eq. (14.8), it is seen that the two different concepts for the number of events n produce equations of a similar form. However, Eq. (14.11) includes the zero crossing term $\overline{\nu}_0$, while Eq. (14.8) involves the frequency bandwidth B. Furthermore, even if the signal had a uniform power spectrum from DC to a high frequency cutoff f_b (for this case, $\overline{\nu}_0 = 1.15B$), the constant coefficient would be slightly different in the two equations.

For the moment, the number of events n will be considered to be a function of some indefinite parameter X with an indefinite coefficient C_1 as follows.

$$n = C_1 \, \Delta x \, \hat{p}(x) \, XT \quad ; \quad X = B \text{ or } \overline{\nu}_0 \qquad (14.12)$$

(c) <u>Population Variance</u>

In Eq. (14.6), the population variance σ_p^2 may be thought of as the variance of a probability density estimate based upon one event. That is, $\sigma_p^2 = \left[\text{Var } \hat{p}(x) \text{ for } n = 1\right]$.

The term σ_p^2 is a function of the variability in crossings of the amplitude range $(x, x+\Delta x)$, as well as the variability in the time τ spent by the signal in the interval Δx during each crossing. It does not appear that this variability can be readily defined from theoretical considerations alone.

In Ref. [1], it is assumed that the standard deviation of the population, σ_p, for any amplitude x is approximately equal to the probability density at that amplitude, p(x). Then $\sigma_p^2 \approx p^2(x)$. However, this value for σ_p^2 is clearly conservative (high). To illustrate this point, assume for any amplitude x that the single event probability density estimates $\hat{p}_i(x)$ which make up the population are symmetrically and uniformly distributed about the true probability density p(x). Since $\hat{p}_i(x)$ can never be negative, the limits of a symmetrical distribution would be zero and 2 p(x). For this simple case, $\sigma_p^2 = (1/3)\, p^2(x)$. Thus, based upon the above qualitative consideration, σ_p^2 is considered to be given by

$$\sigma_p^2 = C_2\, p^2(x) \quad ; \quad C_2 < 1 \tag{14.13}$$

In summary, substituting Eqs. (14.13) and (14.12) into Eq. (14.6), the following expression is obtained for the variance of a probability density estimate.

$$\sigma^2_{\hat{p}(x)} = \frac{\sigma_p^2}{n} = \frac{C_2\, p^2(x)}{C_1 \;\Delta x\, \hat{p}(x)\, XT} \tag{14.14}$$

It is convenient to define the estimation uncertainty in terms of a normalized variance ϵ^2 as follows.

$$\epsilon^2 = \frac{\sigma^2_{\hat{p}(x)}}{p^2(x)} = \frac{C_0}{\Delta x\, \hat{p}(x)\, XT} \quad ; \quad \begin{array}{c} C_0 = C_1/C_2 \\ X = B \text{ or } \overline{\nu}_0 \end{array} \tag{14.15}$$

In Eq. (14.15), the two coefficients, C_1 and C_2, are both indefinite in value, so it is logical to combine them into a single coefficient C_0.

The determination of the coefficient C_0 as well as the correct parameter represented by the term **X** is the primary motivation for the experimental study reported herein.

14.1.3 Application of Estimation Uncertainty

The positive square root of Eq. (14.15) gives a normalized standard deviation for $\hat{p}(x)$, which is often called the normalized standard error. That is,

$$\epsilon = \left[\frac{C_0}{\Delta x\, \hat{p}(x)\, XT}\right]^{1/2} \tag{14.16}$$

It is important to note here that ϵ is a function of the probability density estimate, $\hat{p}(x)$. For each measurement at any amplitude x, a new normalized standard error ϵ must be computed.

From sampling theory, as the number of events n used for an estimate $\hat{p}(x)$ becomes large, the distribution of $\hat{p}(x)$ for any given value of x will approach a normal distribution with a mean value of $p(x)$, regardless of the distribution of the population. The normal approximation should be reasonably sound if n is, say, 10 or larger. For this case, the sampling distribution for an estimate $\hat{p}(x)$ may be considered to be normal with a mean of $p(x)$ and a standard deviation of $\epsilon\, p(x)$. Then, a $(1 - \alpha)$ confidence interval for the true probability density $p(x)$ at any amplitude x will be as follows.

$$\hat{p}(x) - \epsilon\, p(x)\, z_{\alpha/2} \leq p(x) \leq \hat{p}(x) + \epsilon\, p(x)\, z_{\alpha/2} \tag{14.17}$$

Here, $z_{\alpha/2}$ is the normal deviate. From Eq. (14.17), the $(1 - \alpha)$ confidence interval limits for $p(x)$ based on a measured estimate $\hat{p}(x)$ are given by

$$p(x) = \hat{p}(x) \pm \epsilon\, p(x)\, z_{\alpha/2} \tag{14.18}$$

By rearranging terms in Eq. (14.18), the following result is obtained.

$$p(x) = \frac{\hat{p}(x)}{1 \pm \epsilon\, z_{\alpha/2}} \tag{14.19}$$

14.1.4 Frequency Range Considerations

The indefinite parameter X in Eq. (14.15) is believed to be either the frequency bandwidth B, in cycles per second, or the expected number of zero crossings per second $\overline{\nu}_0$ for the signal. In either case, X is related to the spectral composition of the signal. Some discussion of both parameters is warranted.

(a) Frequency Bandwidth

Referring to Eq. (14.8), the bandwidth B is assumed to be ideal with infinitely sharp cutoffs. An equivalent ideal bandwidth may be computed for any real linear filter as discussed in Section 13.1.3. Furthermore, it is shown in Section 16.1.5 that, for most practical applications, the equivalent bandwidth of a random signal with a power spectrum of G(f) is given by

$$B = \frac{1}{G_{max}} \int_0^{\infty} G(f)\, df = \frac{\sigma_x^2}{G_{max}} \qquad (14.20)$$

In words, the equivalent ideal bandwidth is equal to the mean square value of the signal divided by the peak value of the power spectrum. For an arbitrary filter with equivalent ideal cutoff frequencies f_a and f_b, the bandwidth $B = (f_b - f_a)$.

(b) Expected Zero Crossings Per Second

The expected number of zero crossings per second $\overline{\nu}_0$ is a function of the probability density function p(x) and the power spectral density function G(f) for the signal. From Ref. [2], for a Gaussian probability density function, the term $\overline{\nu}_0$ is given by

$$\overline{\nu}_0 = 2 \left[\frac{\int_0^{\infty} f^2 G(f)\, df}{\int_0^{\infty} G(f)\, df} \right]^{1/2} \qquad (14.21)$$

Equation (14.21) is a good approximation for most real physical response signals because a substantial deviation in p(x) from the Gaussian form is required to produce significant changes in the expected number of zero crossings $\overline{\nu}_0$.

Consider the case where the power spectrum G(f) is uniform between lower and upper frequency limits, f_a and f_b, respectively. Equation (14. 21) reduces to

$$\overline{\nu}_0 = 2\left[\frac{f_a^2 + f_a f_b + f_b^2}{3}\right]^{1/2} \tag{14. 22}$$

The validity of Eq. (14. 22) when applied to real analog data is thoroughly substantiated by experiments reported in Section 12. The techniques for computing the effective cutoff frequencies, f_a and f_b, are developed in Section 12. 3. 2.

Two interesting special cases for Eq. (14. 22) should be noted. First, for a power spectrum which is uniform down to DC(f_a = 0), Eq. (14. 22) becomes

$$\overline{\nu}_0 = 1.15 f_b = 1.15B \tag{14. 23}$$

Second, for a power spectrum consisting of a sharp peak at some center frequency f_c, Eq. (14. 22) becomes

$$\overline{\nu}_0 = 2 f_c \tag{14. 24}$$

It is clear from Eq. (14. 22) that for all cases, $\overline{\nu}_0 > B$, since B is given by $(f_b - f_a)$.

14. 1. 5 Averaging Time Considerations

Referring to Eq. (14. 16), the record length T is the total length of the sample amplitude time history made available for analysis. It is assumed that a probability density estimate for each amplitude x is obtained by averaging over the entire available length of data. In other words, in Eq. (14. 16), the averaging time and the record length are ideally the same value, T.

Consider the cases where the averaging time is not equal to the record length T. Let T_1 be the averaging time. The situation is exactly as discussed in Section 13. 1. 3. If $T_1 < T$, the normalized standard error ϵ will be determined by the value of T_1.

If $T < T_1$, the value of ϵ is not reduced below that value obtained using T. That is,

$$\epsilon = \sqrt{\frac{C_0}{\Delta x\, \hat{p}(x)\, XT_1}} \qquad T_1 \leq T \qquad (14.25a)$$

$$\epsilon = \sqrt{\frac{C_0}{\Delta x\, \hat{p}(x)\, XT}} \qquad T < T_1 \qquad (14.25b)$$

For those cases where averaging is accomplished by continuous smoothing of the instantaneous probability density signal with a low pass RC filter, the equivalent averaging time T_1 is given by

$$T_1 = 2K \qquad (14.26)$$

where K is the RC time constant of the low pass filter. Equation (14.26) assumes the signal has been applied for three or four time constants before a reading is taken, and that the time constant K is relatively long compared to the period of the lowest frequency of interest. Substituting Eq. (14.26) into Eq. (14.25), the following result is obtained for RC averaging type instruments.

$$\epsilon = \sqrt{\frac{C_0}{2\,\Delta x\, \hat{p}(x)\, XK}} \qquad K \leq \frac{T}{2} \qquad (14.27a)$$

$$\epsilon = \sqrt{\frac{C_0}{\Delta x\, \hat{p}(x)\, XT}} \qquad \frac{T}{2} \leq K \qquad (14.27b)$$

In Eq. (14.27), ϵ defines the normalized standard error of the continuous estimate at any instant of time.

Additional information on the practical considerations of probability density estimation is presented in Ref. [1], Section 7.5.

14.2 DESIGN OF EXPERIMENTS AND PROCEDURES

14.2.1 General Design and Procedures

The general purpose of these experiments is to obtain an appropriate definition for the statistical uncertainty of probability density estimates from sample records of stationary random signals. To meet this general goal, the experiments are designed for two specific objectives.

(a) To determine and verify the uncertainty expression given in general form by Eq. (14.15). That is, to establish the proper parameter represented by the term X and the appropriate value for the coefficient C_0.

(b) To establish that probability density estimates may be considered normally distributed about the true probability density for any amplitude x.

(a) Determination and Verification of Uncertainty Expression

To establish the specific form of Eq. (14.15), the normalized variance ϵ^2 is experimentally determined by gathering a series of N number of estimates $\hat{p}_v$* for a specific amplitude v and a given set of measurement parameters, Δx, f_a, f_b, and T_1. An empirical value for the coefficient C_0 is then computed two ways. One value is computed assuming the parameter $X = B = (f_b - f_a)$, and another value is computed assuming the parameter $X = \overline{\nu}_0$ as defined in Eq. (14.22). The experiment is repeated for M different sets of values for v, f_a, f_b, and T_1. The instrument employed to measure $\hat{p}_v$ has a fixed amplitude window width Δx, so this parameter cannot be varied. If the form of Eq. (14.15) is valid, the resulting set of values for the coefficient C_0 should be equivalent when determined using one of the two parameters, B or $\overline{\nu}_0$.

*The notation $\hat{p}_v$ means a probability density estimate at a specific amplitude v; that is, $\hat{p}_v = \hat{p}(x)$ for $x = v$.

To be more specific, for each sample of N number of statistically independent estimates $\hat{p}_v$, a sample mean and variance are computed as follows.

$$\bar{p} = \frac{1}{N}\sum_{i=1}^{N}\hat{p}_{vi} \tag{14.28a}$$

$$s^2 = \frac{1}{N}\sum_{i=1}^{N}(\hat{p}_{vi} - \bar{p})^2 = \frac{1}{N}\sum_{i=1}^{N}\hat{p}_{vi}^2 - (\bar{p})^2 \tag{14.28b}$$ *

The expected values for the above sample mean and variance are the probability density, p_v, and the variance of the estimate, $\sigma^2_{\hat{p}_v}$. That is,

$$E\left[\bar{p}\right] = p_v$$

$$E\left[s^2\right] = \left(\frac{N-1}{N}\right)\sigma^2_{\hat{p}_v} \approx \sigma^2_{\hat{p}_v} \quad \text{for large N} \tag{14.29}$$

From Eq. (14.15), the coefficient C_0 is estimated as follows.

$$\hat{C}_0 = \frac{s^2}{\bar{p}^2}\left[\Delta x\, \bar{p}\, X T_1\right] \tag{14.30}$$

Thus, two different empirical values for C_0 are obtained as follows.

$$\hat{C}_0 = \frac{s^2 \Delta x\, B T_1}{\bar{p}} \tag{14.31a}$$

$$\hat{C}_0 = \frac{s^2 \Delta x\, \bar{\nu}_0\, T_1}{\bar{p}} \tag{14.31b}$$

The experiment is repeated M times with different values for v, f_a, f_b, and T_1 to obtain M number of pairs of empirical values for C_0.

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*A biased expression for s^2 is employed here so that all statistical procedures to follow will be consistent with procedures outlined in Ref. [1].

(b) Verification that Estimates are Normally Distributed

The general procedure is to gather a large number of estimates, $\hat{p}_v$, for a specific amplitude v and a given set of measurement parameters, Δx, f_a, f_b, and T_1. The large sample is then tested for normality by a "chi-squared goodness of fit" test.

14.2.2 Detailed Test Procedure

The random signal source used in these experiments has an approximately Gaussian probability density function and uniform power spectrum over the frequency range of interest. The frequency range of the signal is limited either by a band pass filter with variable lower and upper half power cutoff frequencies of f_{ahp} and f_{bhp}, respectively, or by a narrow band pass filter with a sharply defined bandwidth B on a center frequency of f_c. The rms voltage level of the band-limited signal is set at one volt for all experiments. Probability density estimates are obtained using a commercial analyzer which has a fixed amplitude window of $\Delta x = 0.1$ volt, and which produces a continuous probability density estimate by averaging with an equivalent low pass RC filter having several different time constant selections available. The continuously averaged estimate is recorded to obtain a plot of $\hat{p}_v$ versus time. Specific values for $\hat{p}_v$ are obtained by reading values from the continuous plot at equally spaced time intervals.

The time interval between readings is selected to be at least 4K seconds to assure the $\hat{p}_v$ values are statistically independent. The details of all instruments, test set-up, and calibrations are presented in Section 14.3.

(a) Determination and Verification of Uncertainty Expression

A total of N = 61 statistically independent estimates $\hat{p}_v$ are gathered for each of M = 18 different sets of values for K, v, f_{ahp}, and f_{bhp}, as follows

A. $\Delta x = 0.100$ volts; v = 0 volts; $f_{ahp} = 100$ cps; $f_{bhp} = 600$ cps.

A-1. K = 0.515 seconds ($T_1 = 1.03$ seconds)

A-2. K = 0.202 seconds ($T_1 = 0.404$ seconds)

A-3. $K = 0.109$ seconds ($T_1 = 0.218$ seconds)

A-4. $K = 0.040$ seconds ($T_1 = 0.080$ seconds)

B. $\Delta x = 0.100$ volts; $K = 0.109$ seconds ($T_1 = 0.218$ seconds); $f_{ahp} = 100$ cps; $f_{bhp} = 600$ cps

B-1. $v = 0.50$ volts

B-2. $v = 1.00$ volts

B-3. $v = 1.50$ volts

B-4. $v = 2.00$ volts

B-5. $v = 2.50$ volts

C. $\Delta x = 0.100$ volts; $v = 0$ volts; $K = 0.040$ seconds ($T_1 = 0.080$ seconds)

C-1. $f_{ahp} = 1000$ cps ; $f_{bhp} = 2000$ cps

C-2. $f_{ahp} = 2000$ cps ; $f_{bhp} = 4000$ cps

C-3. $f_{ahp} = 4000$ cps ; $f_{bhp} = 7000$ cps

C-4. $f_{ahp} = 7000$ cps ; $f_{bhp} = 12000$ cps

D. $\Delta x = 0.100$ volts; $v = 0$ volts; $K = 0.515$ seconds ($T_1 = 1.03$ seconds); $B = 56$ cps.

D-1. $f_c = 100$ cps

D-2. $f_c = 500$ cps

D-3. $f_c = 1000$ cps

D-4. $f_c = 4000$ cps

D-5. $f_c = 7000$ cps

The relatively short time constants in A are employed to obtain a sufficient variability in the estimates to minimize readout errors. The maximum amplitude level in B is limited to 2.5 volts (2.5 times the rms level) because the probability density signals at higher amplitudes are too small to be read out and interpreted with reasonable accuracy for the test set-up used for these experiments. The frequency ranges in C are selected to cover the frequency range of predominate acoustic and vibration response in modern flight vehicles. The narrow frequency bandwidths with the wide range of center frequencies in D are selected to represent a narrow band vibration response for a lightly damped resonant structure.

(b) Verification that Estimates are Normally Distributed

A total of N = 183 statistically independent estimates $\hat{p}_v$ are gathered with the following test parameters.

$$\Delta x = 0.100 \text{ volts}$$
$$v = 0 \text{ volts}$$
$$K = 0.109 \text{ seconds}$$
$$f_{ahp} = 100 \text{ cps}$$
$$f_{bhp} = 600 \text{ cps}$$

The equivalent number of events for each estimate is $n \approx 9$ using Eq. (14.8), or $n \approx 17$ using Eq. (14.11).

14.2.3 Statistical Hypothesis Tests

(a) Determination and Verification of Uncertainty Expression

In order to test the equivalence of the experimentally determined values of C_0, it is necessary to define the sampling distribution for $\hat{C}_0$. From Eq. (14.31), it is seen that $\hat{C}_0$ is a function of five parameters, all of which are in reality random variables. The terms s^2 and $\bar{p}$ are sample statistics which, of course, involve a probable sampling error. The parameters Δx, B or $\overline{\nu}_0$, and T_1 also have some variability associated with their determinations. The true sampling distribution for $\hat{C}_0$ is a function of all these variabilities. The determination of this sampling distribution would clearly be a difficult problem.

As was done for the study of power spectra estimates in Section 13, it is assumed here that the variability of the estimated values $\hat{C}_0$ is due only to the variability of the term s^2. That is, it is assumed that $\bar{p} = p_v$ and that the values for Δx, B or $\overline{\nu}_0$, and T_1 are exact. The effect of this assumption is discussed later.

The sample variance s^2 has a distribution associated with a chi-squared distribution as follows.

$$\frac{s^2}{\sigma^2_{\hat{p}_v}} \sim \frac{\chi^2}{61} \qquad (14.32)$$

where "$\sim$" means "distributed as", and χ^2 is chi-squared with $(N - 1) = 60$ degrees of freedom. By substituting the relationships from Eq. (14. 30) into Eq. (14. 32), the following result is obtained

$$\frac{\hat{C}_0}{(\sigma^2_{\hat{p}_v}/\bar{p}^2)(\Delta x\ \bar{p}\ XT)} \sim \frac{\chi^2}{61} \tag{14. 33}$$

If all terms in the denominator of Eq. (14. 33) are considered expected values, the denominator is equal to the expected value for the coefficient C_0. Then,

$$\frac{\hat{C}_0}{C_0} \sim \frac{\chi^2}{61} \tag{14. 34}$$

The problem remaining is to determine if the collection of experimental values for $\hat{C}_0$ are equivalent to the same constant coefficient C_0 when the parameter X is either B or $\bar{\nu}_\theta$. If the values $\hat{C}_0$ are equivalent for either case, the sampling distribution for $\hat{C}_0$ will be as given in Eq. (14. 34), and will have an expected variance as follows.

$$\sigma^2_{\hat{C}_0} = \text{Var}\left(\frac{C_0\chi^2}{61}\right) = \left(\frac{C_0}{61}\right)^2 \text{Var}\ \chi^2 \tag{14. 35}$$

The variance of a χ^2 distribution is two times the number of degrees of freedom, or 120 for χ^2 with 60 degrees of freedom. Then,

$$\sigma^2_{\hat{C}_0} = 120\left(\frac{C_0}{61}\right)^2 = 0.0322\, C_0^2 \tag{14. 36}$$

Now, if the M = 18 values for $\hat{C}_0$ are not equivalent, the variance of the experimental values will clearly be greater than predicted by Eq. (14. 36). Then, the values can be tested for equivalence by comparing $\sigma^2_{\hat{C}_0}$ with the estimated variance $s^2_{\hat{C}_0}$, computed as follows.

$$s^2_{\hat{C}_0} = \frac{1}{M}\sum_{i=1}^{M}(\hat{C}_{0i} - C_0)^2 = \frac{1}{18}\sum_{i=1}^{18}\hat{C}^2_{0i} - C_0^2 \tag{14. 37}$$

For both Eqs. (14. 36) and (14. 37), C_0 is considered to be the average of the 18 values for $\hat{C}_0$.

Let it be hypothesized that all experimental values, $\hat{C}_0$, are equivalent. Then, the hypothesis H_0 becomes

$$H_0 \;:\; s^2_{\hat{C}_0} = \sigma^2_{\hat{C}_0} \tag{14. 38}$$

If the hypothesis is true, the following probability statement applies.

$$\text{Prob}\left[\frac{s^2_{\hat{C}_0}}{\sigma^2_{\hat{C}_0}} \leq \frac{\chi^2_\alpha}{18}\right] = (1 - \alpha) \tag{14. 39}$$

where χ^2_α is chi-squared with $(M - 1) = 17$ degrees of freedom for the α level of significance. Thus, to test H_0 at the $\alpha = 0.05$ level of significance, the region of acceptance is

$$\frac{s^2_{\hat{C}_0}}{\sigma^2_{\hat{C}_0}} \leq 1.53 \tag{14. 40}$$

If the variance ratio is within the noted limit, H_0 is accepted and the experimentally computed values, $\hat{C}_0$, are accepted as equivalent. If the variance ratio falls outside the noted limit, H_0 is rejected and there is reason to suspect that the values for $\hat{C}_0$ are not equivalent.

The probability of a Type I error for a test of H_0 is, of course, $\alpha = 0.05$. The probability of a Type II error is $\beta = 0.05$ for detecting a ratio of 3:1 for the true variance estimated by $s^2_{\hat{C}_0}$ and the predicted variance $\sigma^2_{\hat{C}_0}$. The Type II error is determined directly from the Operating Characteristic (OC) curve for a one-sided (upper tail) χ^2 test at $\alpha = 0.05$ for 18 samples. This OC curve is presented in Figure 6. 17 of Ref. [3].

It should be noted that the test for equivalence may be applied to any group of estimates $\hat{C}_0$ individually. For example, the value of $\hat{C}_0$ obtained for the experiments of Test Procedure A may be tested for equivalence to determine if $\hat{C}_0$ is effected by changes in the averaging time. Such individual tests for equivalence are performed as required by the results. Of course, a different Type II error probability will be associated with such individual tests using different sample sizes.

Consider now the assumption employed to arrive at Eq. (14.34); namely, $\bar{p}$, Δx, X, and T_1 are exact values. The obvious effect of this assumption is to reduce the predicted variance of the random variable $\hat{C}_0$. In other words, the variability of $\hat{C}_0$ is actually greater than predicted by Eq. (14.34), and the region of acceptance for a test of H_0 at the $\alpha = 0.05$ level of significance is actually greater than predicted by Eq. (14.40). The net result is that the probability of making a Type I or Type II error for the test of H_0 is larger than the $\alpha = \beta = 0.05$ value. Thus, it can only be said that the probability of a Type I or Type II error for the test of H_0 in Eq. (14.38) is <u>at least</u> 5%.

(b) Verification that Estimates are Normally Distributed

The 183 estimates $\hat{p}_v$ are tested for normality by applying a "chi-squared goodness of fit" test, as detailed in Section 5.3.2 of Ref. [1]. A null hypothesis is established as follows.

$$H_0 \ : \ \hat{p}_v \sim \text{normally} \tag{14.41}$$

The 183 values of $\hat{p}_v$ are divided into k = 16 class intervals. The statistic χ^2 is computed as follows.

$$\chi^2 = \sum_{i=1}^{k} \frac{(f_i - F_i)^2}{F_i} \tag{14.42}$$

where f_i is the observed frequency in the ith class and F_i is the expected frequency in the ith class. The hypothesis H_0 requires that $\chi^2 < \chi_\alpha^2$, where χ_α^2 is the actual value of chi-squared for k - 3 = 13 degrees of freedom. The test of H_0 is performed at the $\alpha = 5\%$ level of significance, $\chi^2_{0.05} = 22.36$, so

$$\chi^2 < 22.36 \tag{14.43}$$

If the computed value χ^2 is less than 22.36, H_0 is accepted and the various values for $\hat{p}_v$ are considered normally distributed. If χ^2 is greater than 22.36, H_0 is rejected. This means that there is reason to doubt that the estimates $\hat{p}_v$ are normally distributed.

The probable Type I error is $\alpha = 0.05$. Because non-normality can occur in an unlimited number of ways, there is no definition for a meaningful Type II error.

14.3 INSTRUMENTATION

14.3.1 Instruments and Test Set-Up

The laboratory instruments employed for the experiments are listed in Table 14.1. A block diagram for the test set-up is illustrated in Figure 14.2. Except for the random noise generator, all instruments are the property of the Norair Division of Northrop Corporation, and were in current calibration at the time of the experiments.

The random noise generator (Item A) is used as the source of a signal which is considered in these experiments to be a stationary random signal. The instrument generates noise with an approximately Gaussian amplitude probability density function and an approximately uniform power spectral density function over a frequency range from less than 100 cps to over 12000 cps (the frequency range of interest in these experiments). The probability density function measured for the noise generator is presented in Figure 14.3. The power spectrum measured for the noise generator is presented in Figure 14.4.

The sine wave generator (Item B) is used for calibration purposes as discussed in Sections 14.3.3 and 14.3.4. The frequency counter (Item C) is used to determine the frequency of the sine wave calibration signals with an accuracy of $\pm$ 1 cps. The voltmeter (Item D) is used to measure the output rms voltage level of the random noise generator, or the sine wave generator, for reference purposes.

The variable band pass filter (Item E-1) is used to limit the frequency range of the random signal for relatively broad bandwidths, as required for Test Procedures A, B, and C in Section 14.2.2. The narrow band pass filter (Item E-2) is used to limit the frequency range of the random signal for relatively narrow bandwidths, as required for Test Procedure D in Section 14.2.2. The characteristics and calibration of these filters are discussed in Section 14.3.2.

The voltage amplifier (Item F) is used to increase the voltage output of the band pass filter to a level required for proper operation of the probability density analyzer (at least one volt rms). The voltmeter (Item G) is used to measure the rms voltage level of the signal to be investigated.

Item	Description	Manufacturer	Model No.	Serial No. *
A	Random Noise Generator	General Radio Co.	1390A	937
B	Audio Oscillator	Hewlett-Packard Co.	202D	67117
C	Universal EPUT and Timer	Berkeley Division	7350	79528
D	True RMS Voltmeter	Ballantine Laboratories, Inc.	320	80798
E-1	Variable Band Pass Filter	Krohn-Hite Instrument Co.	330A	56860
E-2	Narrow Band Pass Filter	Technical Products Co.	**	
F	Voltage Amplifier	Computer Engineering	A1-233-B	PR3877-4
G	True RMS Voltmeter	B & K Instruments, Inc.	2409R	PR13147-1
H	Probability Density Analyzer	B & K Instruments, Inc.	160	PR10011-1
I	Strip Chart Recorder	Brush Instruments	Mark II	270822

*Except for Item A, the serial numbers refer to Northrop Corporation identification tags.

**This filter is incorporated in TPC power spectral density analyzer. See Section 13. 3, Item E.

Table 14. 1 Instruments Employed for the Experiments

The probability density analyzer (Item H) is used to obtain probability density estimates at various amplitude levels and with various averaging times, as discussed in Section 14. 3. 3. The strip chart recorder (Item I) is used to record probability density time history samples for the signal to be investigated.

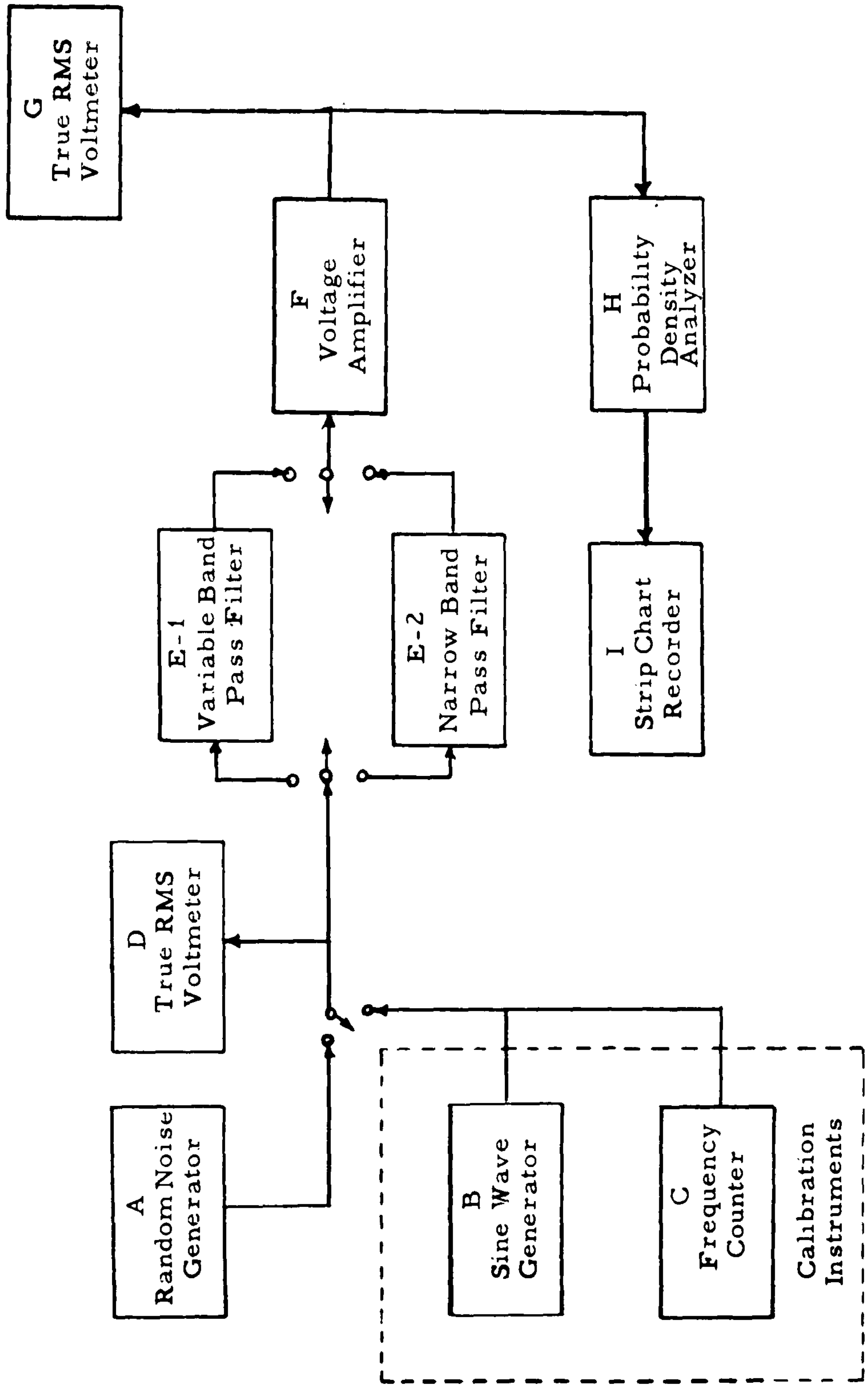

Figure 14.2 Block Diagram of Test Set-Up

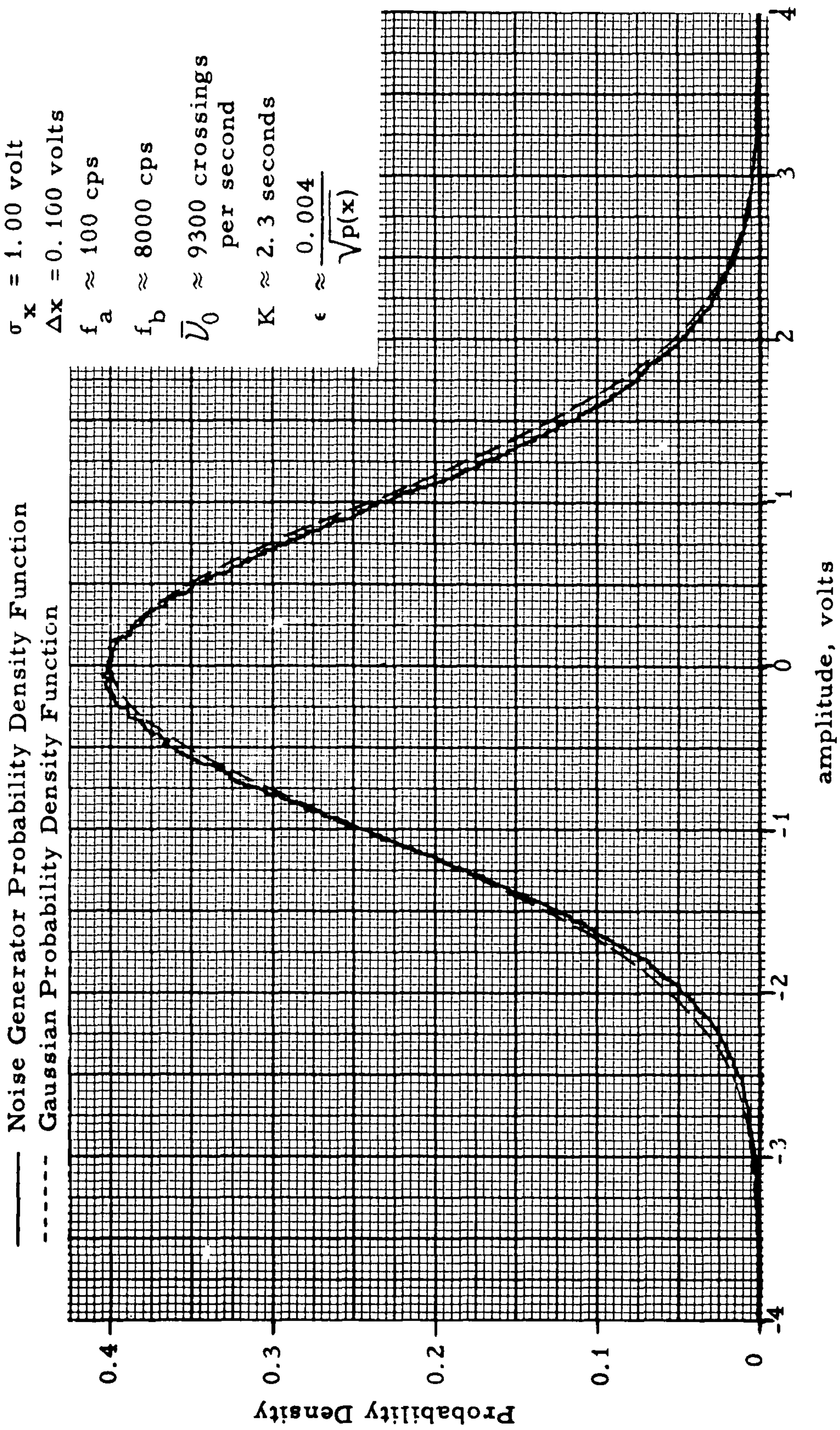

Figure 14.3 Measured Probability Density Function for Random Noise Generator

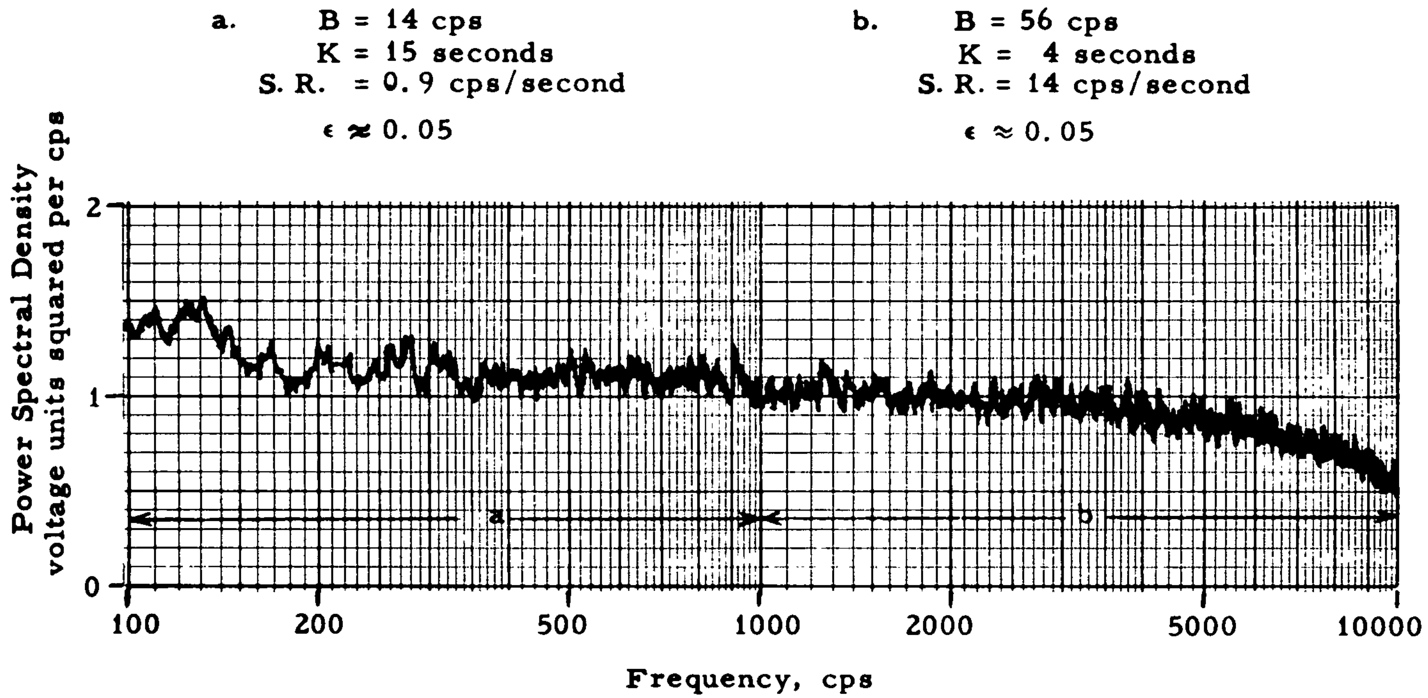

Figure 14.4 Measured Power Spectral Density Function for Random Noise Generator

14.3.2 Characteristics of Band Pass Filters

To determine the value of B and $\overline{\nu}_0$ for the band-limited signals being studied, it is necessary to compute the lower and upper effective cutoff frequencies for the band pass filters. The procedure for computing f_a and f_b is detailed in Section 12.3.2. In general, for a band pass filter with a frequency response function H(f) which has a maximum value H_{max} at some frequency f_c, the bandwidth $B = (f_b - f_a)$ where the effective cutoff frequencies are given by

$$f_a = f_c - \int_0^{f_c} \left| \frac{H(f)}{H_{max}} \right|^2 df$$
$$f_b = f_c + \int_{f_c}^{\infty} \left| \frac{H(f)}{H_{max}} \right|^2 df \qquad (14.44)$$

(a) Variable Band Pass Filter, Item E-1

The variable band pass filter used for Test Procedures A, B, and C is the same type of filter (Krohn-Hite Model 330) that was carefully calibrated for several different cutoff frequencies in Section 12.3.2. The results there indicate the effective cutoff frequencies for the filter in terms of the lower and upper half power points, f_{ahp} and f_{bhp}, respectively, may be considered to be as follows.

$$f_a = 0.95 f_{ahp} \quad ; \quad f_b = 1.05 f_{bhp} \qquad (14.45)$$

To confirm that Eq. (14.45) is applicable to the specific band pass filter employed in these experiments, the cutoff characteristics of the filter have been carefully determined for the case where f_{ahp} = 100 cps and f_{bhp} = 600 cps. The results are presented in Figure 14.5. A graphical integration of the quantity $|H(f)|^2$ in Figure 14.5 produces approximately the same results presented in Eq. (14.45).

(b) Narrow Band Pass Filter, Item E-2

The narrow band filter employed for Test Procedure D is the 56 cps bandwidth filter that was calibrated in Section 13.3.2. The results there indicate the cutoff characteristics are sufficiently sharp to assume the

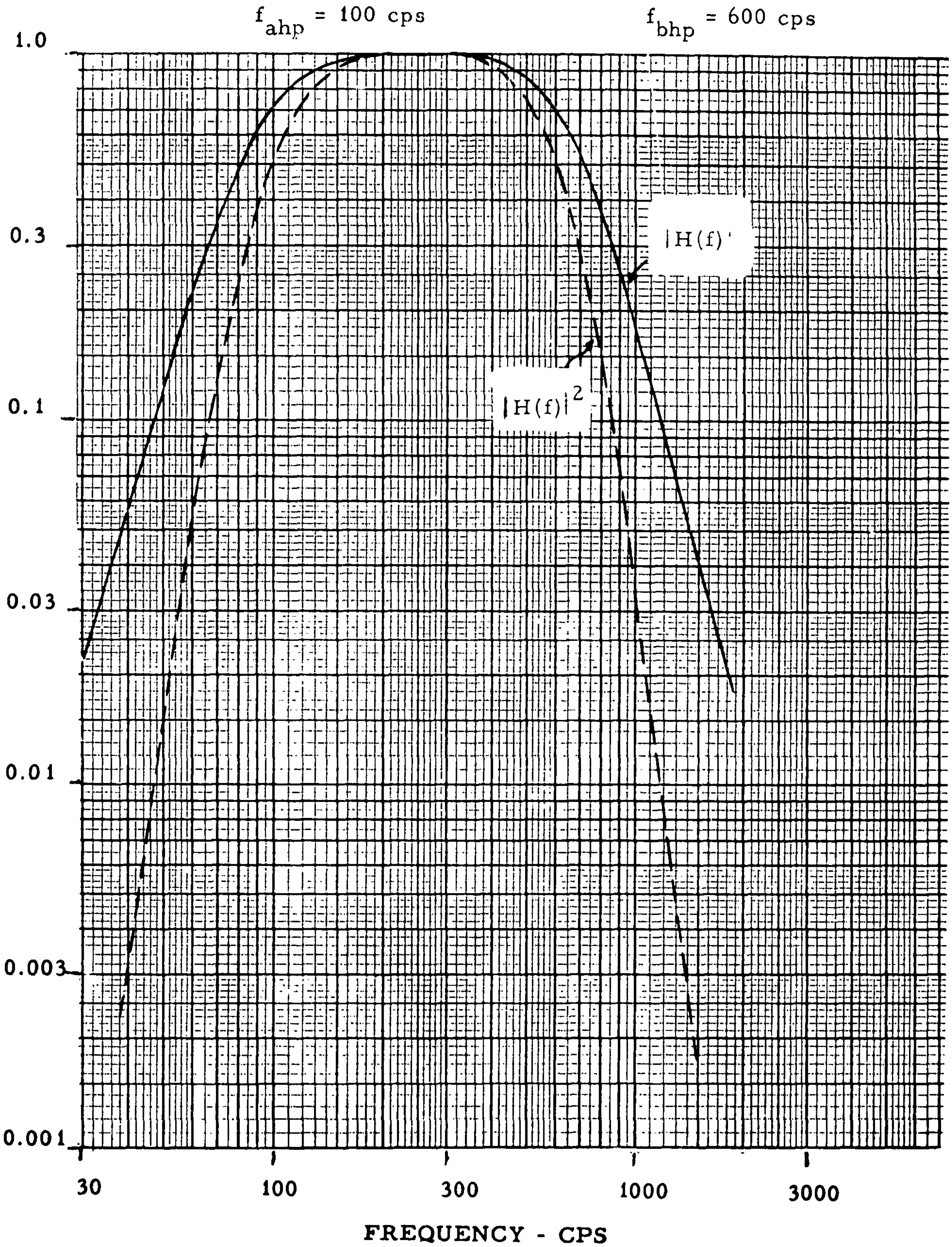

Figure 14.5 Variable Band Pass Filter Characteristics

the filter has an ideal rectangular response shape. Thus, the effective cutoff frequencies are $f_c \pm (1/2)B$. That is,

$$
\begin{aligned}
f_a &= f_c - 28 \\
f_b &= f_c + 28
\end{aligned}
\qquad (14.46)
$$

14.3.3 <u>Probability Density Analyzer</u>

The amplitude window width Δx and the averaging time $T_1 = 2K$ are characteristics of the probability density analyzer. Although the manufacturer of the analyzer publishes nominal values for these parameters, they are determined independently for these experiments as follows.

(a) <u>Amplitude Window Width</u>

A signal with a zero rms level (no signal at all) theoretically has a probability density function consisting of a unity area delta function at the zero amplitude level. That is,

$$
p(x) = \delta(x) \begin{cases} 1 \text{ for } x = 0 \\ 0 \text{ for } x \neq 0 \end{cases} \qquad (14.47)
$$

The window width Δx for the analyzer is readily established by moving the window in amplitude through the zero level with no signal applied to the analyzer. When the zero level is included inside the window Δx, the probability density meter indication is pegged at full scale. When the zero level is not included inside the window Δx, the probability density meter indication is zero. By this procedure, the window width Δx has been established to be 0.100 volts limited by the probable errors in the calibration and reading of the amplitude level scale. These errors are estimated to be less than +5% of Δx.

(b) <u>Averaging Time Constants</u>

The probability density analyzer is equipped with seven different averaging time constant selections which are identified as A through G on a selector dial. Only four of the selections (C, D, E, and F) are used in these experiments. The value for the time constant K associated with each of the four averaging time selections is established by the following procedure.

Referring to Figure 14. 2, a sinusoidal voltage signal from Item B is applied to the test set-up and read out as a probability density measurement at Item I. Since the applied signal is periodic, there is no statistical uncertainty in the resulting probability density indications. Item I is operated with a high chart speed (125 mm/sec) so that the response time characteristics of the test set-up are accurately defined. The time constant K is determined by measuring the time required for a probability density indication to rise from zero to 63. 4% of the final steady state value when the sinusoidal signal is applied instantaneously, or the time required for the probability density indication to fall 63. 4% from the steady state value when the sinusoidal signal is removed instantaneously. This is done at several different frequencies to obtain at least 10 measurements of K for each selector setting of interest on Item H.

Using the above procedure, the time constants associated with the selector settings C, D, E, and F have been determined to be as follows.

C. $K = 0.515 \pm 0.009$ seconds

D. $K = 0.202 \pm 0.007$ seconds

E. $K = 0.109 \pm 0.003$ seconds

F. $K = 0.040 \pm 0.0014$ seconds

The tolerance figures shown for each value of K are the standard deviations of the measurements.

It should be mentioned that the accuracy of the values determined for K is a function of the accuracy of the paper speed control of the recorder, Item I. The accuracy of the recorder paper speed has been checked by directly recording sine waves of various frequencies determined to ± 1 cps using Item C. The results indicate the error in the indicated paper speed to be insignificant.

14.3.4 Other Instrument Evaluations and Calibration Errors

There are, of course, numerous possible errors associated with the calibrations of all the instruments employed for the experiments. However, most of these calibration errors are not really significant in terms of the end data being sought; namely, probability density estimates $\hat{p}_v$ as a function of Δx, K, v, f_a, and f_b. The determinations and accuracies of Δx and K are discussed in Section 14.3.3. The value of v does not enter into Eq. (14.30), so its accuracy is of no direct importance.

For the case of the variable band pass filter (Item E-1), the possible error in the values for f_a and f_b is the result of several factors including the determination of the half power points of the filter, the calculation of the effective cutoff frequencies as compared to the half power points, and drift in the filter characteristics. Referring to Figure 14.2, the half power points for Item E-1 are determined by reading the frequency from Item C when the filter response is 3 db down from the maximum response, as read from Item G, for a sine wave applied at a constant input voltage, as read from Item D. The indirect sources of error are then the lack of a perfect frequency response in the two voltmeters (Items D and G), the $\pm$ 1 cps accuracy of the frequency counter (Item C), and observational errors. It is believed that the total effect of these factors on the determination of the half power point frequencies is less than $\pm$ 1% for these experiments. The half power points are carefully checked immediately before each experiment. Since a single record run is never greater than two minutes, the effect of filter drift is considered negligible. The computation of f_a and f_b for any given half power points is assessed to be within $\pm$ 2%. Thus, the maximum error in the values f_a and f_b is believed to be less than $\pm$ 3%.

For the case of the narrow band pass filter (Item E-2), it is believed that the determination of the bandwidth B = 56 cps is accurate to $\pm$ 2% based upon repeated measurements.

The possible error in the determination of p_v involves a number of factors including sensitivity drift in any or all instruments in the test set-up, the calibration accuracy of the probability density analyzer

(Item H), the error in establishing the desired rms level for the input random signal, and observational errors. Drift is minimized by permitting the instruments to warm up for several hours before performing an experiment, and by checking all pertinent calibrations before each individual record run. The probability density analyzer is calibrated prior to each experiment by the various devices incorporated in the instrument. The calibration is checked by applying a sine wave from Item B and comparing the analyzer probability density indications with the theoretically probability density function for a sine wave. The random signal is then applied with the same rms level as the calibrating sine wave. The rms level of the signal is established using Item G. There are uncertainty fluctuations in the reading of Item G, and some mental averaging of these fluctuations is required to adjust the input voltage to the desired level. The readout observational errors are indicated by the number of significant figures recorded for the values of $\hat{p}_v$. The last figure in the $\hat{p}_v$ values in Section 14.4 is an estimate. Based upon all these considerations, the accuracy of the values for $\hat{p}_v$ are believed to be approximately ±5%.

14.4 RESULTS OF EXPERIMENTS

14.4.1 Probability Density Data

The 61 probability density estimates gathered for each of the 18 experiments designed to determine and verify the uncertainty expression, Eq. (14.15), are presented in Tables 14.2 through 14.5. The test designations refer to test procedures defined in Section 14.2. The 183 probability density estimates gathered for the normality test are presented in Table 14.6. The first 61 of these values are the same data as given under A-3 in Table 14.2.

A picture of some typical test data is shown in Figure 14.6.

14.4.2 Frequency Bandwidth and Zero Crossing Data

The effective low and high frequency band pass limits, the equivalent ideal frequency bandwidth and the expected number of zero crossings per second associated with the frequency limits are presented in Table 14.7. The effective cutoff frequencies are determined as discussed in Section 14.3.2. The expected number of zero crossings per second are determined using Eq. (14.22).

14.4.3 Sampling Statistics for Probability Density Estimates

The sample mean and variance for the 61 probability density estimates gathered for each of the 18 experiments are presented in Table 14.8. The sample mean and variance are computed using Eq. (14.28).

Probability Density Estimates p_v for Time Constants of K Seconds

rms level = 1.00 volt; Δx = 0.100 volts; v = 0 volts; f_{ahp} = 100 cps; f_{bhp} = 600 cps

Estimate Number	A-1 K = 0.515	A-2 K = 0.202	A-3 K = 0.109	A-4 K = 0.040	Estimate Number	A-1 K = 0.515	A-2 K = 0.202	A-3 K = 0.109	A-4 K = 0.040
1	0.392	0.380	0.385	0.440	31	0.395	0.408	0.485	0.405
2	0.410	0.390	0.390	0.368	32	0.403	0.423	0.422	0.335
3	0.413	0.435	0.358	0.322	33	0.395	0.390	0.340	0.362
4	0.409	0.387	0.418	0.427	34	0.405	0.350	0.334	0.380
5	0.421	0.375	0.379	0.319	35	0.400	0.401	0.401	0.321
6	0.400	0.353	0.344	0.480	36	0.423	0.390	0.434	0.428
7	0.382	0.408	0.365	0.361	37	0.420	0.362	0.435	0.378
8	0.398	0.360	0.368	0.350	38	0.397	0.393	0.405	0.508
9	0.392	0.400	0.360	0.352	39	0.360	0.403	0.362	0.392
10	0.390	0.413	0.348	0.353	40	0.358	0.395	0.342	0.547
11	0.372	0.398	0.418	0.396	41	0.392	0.427	0.380	0.490
12	0.381	0.365	0.437	0.439	42	0.368	0.421	0.370	0.371
13	0.387	0.430	0.442	0.349	43	0.381	0.342	0.430	0.370
14	0.405	0.385	0.378	0.320	44	0.439	0.398	0.290	0.490
15	0.383	0.398	0.425	0.380	45	0.421	0.432	0.448	0.420
16	0.400	0.359	0.422	0.478	46	0.410	0.368	0.419	0.445
17	0.396	0.370	0.420	0.422	47	0.398	0.365	0.378	0.290
18	0.368	0.390	0.357	0.367	48	0.395	0.437	0.363	0.680
19	0.382	0.422	0.401	0.398	49	0.392	0.415	0.402	0.510
20	0.399	0.420	0.551	0.570	50	0.399	0.360	0.402	0.560
21	0.400	0.379	0.362	0.431	51	0.376	0.398	0.439	0.382
22	0.395	0.329	0.470	0.355	52	0.402	0.392	0.379	0.387
23	0.420	0.380	0.400	0.462	53	0.411	0.412	0.377	0.352
24	0.420	0.411	0.348	0.481	54	0.420	0.400	0.377	0.340
25	0.378	0.362	0.452	0.360	55	0.387	0.440	0.360	0.416
26	0.361	0.420	0.365	0.363	56	0.390	0.380	0.408	0.409
27	0.405	0.443	0.343	0.339	57	0.361	0.372	0.330	0.393
28	0.400	0.368	0.390	0.380	58	0.360	0.398	0.456	0.300
29	0.380	0.379	0.391	0.302	59	0.400	0.400	0.420	0.423
30	0.378	0.380	0.385	0.379	60	0.383	0.380	0.365	0.334
					61	0.372	0.421	0.360	0.358

Table 14.2 Probability Density Estimates for Different Averaging Times

Probability Density Estimates $\hat{p}_v$ for Amplitude Levels of v Volts

rms level = 1.00 volt; Δx = 0.100 volts; K = 0.109 seconds; f_{ahp} = 100 cps; f_{bhp} = 600 cps

Estimate Number	B-1 v =0.50	B-2 v=1.00	B-3 v=1.50	B-4 v=2.00	B-5 v=2.50	Estimate Number	B-1 v=0.50	B-2 v=1.00	B-3 v=1.50	B-4 v=2.00	B-5 v=2.50
1	0.415	0.216	0.112	0.037	0.0085	31	0.278	0.186	0.092	0.046	0.0213
2	0.362	0.304	0.111	0.040	0.0403	32	0.370	0.222	0.105	0.058	0.0200
3	0.308	0.298	0.090	0.038	0.0032	33	0.360	0.246	0.114	0.064	0.0138
4	0.313	0.257	0.092	0.050	0.0159	34	0.285	0.249	0.102	0.050	0.0165
5	0.259	0.235	0.110	0.025	0.0360	35	0.325	0.248	0.132	0.031	0.0060
6	0.315	0.220	0.149	0.039	0.0060	36	0.348	0.239	0.124	0.040	0.0156
7	0.291	0.206	0.120	0.057	0.0085	37	0.330	0.219	0.106	0.037	0.0380
8	0.312	0.230	0.115	0.048	0.0138	38	0.321	0.209	0.091	0.052	0.0085
9	0.320	0.190	0.122	0.072	0.0180	39	0.406	0.279	0.116	0.036	0.0152
10	0.359	0.234	0.110	0.053	0.0007	40	0.320	0.250	0.179	0.058	0.0051
11	0.332	0.201	0.144	0.065	0.0025	41	0.302	0.200	0.105	0.036	0.0089
12	0.341	0.200	0.114	0.036	0.0090	42	0.363	0.252	0.120	0.046	0.0211
13	0.334	0.208	0.100	0.036	0.0160	43	0.338	0.249	0.098	0.046	0.0028
14	0.355	0.215	0.100	0.045	0.0101	44	0.311	0.221	0.110	0.041	0.0130
15	0.291	0.269	0.133	0.071	0.0247	45	0.283	0.235	0.125	0.070	0.0056
16	0.322	0.226	0.130	0.076	0.0048	46	0.360	0.186	0.114	0.042	0.0141
17	0.335	0.197	0.138	0.067	0.0145	47	0.321	0.258	0.100	0.027	0.0070
18	0.335	0.196	0.100	0.029	0.0130	48	0.370	0.260	0.144	0.027	0.0178
19	0.298	0.276	0.092	0.073	0.0100	49	0.342	0.194	0.119	0.060	0.0135
20	0.319	0.206	0.123	0.040	0.0248	50	0.320	0.221	0.165	0.042	0.0150
21	0.365	0.201	0.145	0.039	0.0175	51	0.322	0.228	0.143	0.033	0.0195
22	0.375	0.250	0.127	0.032	0.0155	52	0.364	0.251	0.126	0.022	0.0163
23	0.280	0.241	0.101	0.067	0.0100	53	0.310	0.210	0.108	0.044	0.0080
24	0.353	0.257	0.132	0.060	0.0063	54	0.413	0.205	0.114	0.054	0.0022
25	0.295	0.207	0.150	0.038	0.0162	55	0.370	0.220	0.153	0.048	0.0161
26	0.302	0.270	0.100	0.024	0.0230	56	0.339	0.221	0.119	0.041	0.0091
27	0.323	0.256	0.150	0.030	0.0079	57	0.380	0.252	0.108	0.025	0.0350
28	0.391	0.200	0.118	0.041	0.0105	58	0.387	0.232	0.114	0.045	0.0062
29	0.330	0.197	0.090	0.039	0.0305	59	0.332	0.264	0.144	0.079	0.0065
30	0.360	0.239	0.157	0.040	0.0235	60	0.330	0.211	0.127	0.079	0.0109
						61	0.379	0.210	0.137	0.037	0.0075

Table 14.3 Probability Density Estimates for Different Amplitude Levels

Probability Density Estimates $\hat{p}_v$ for Frequency Limits f_{ahp} cps and f_{bhp} cps

rms level = 1.00 volt; Δx = 0.100 volts; v = 0 volts; K = 0.40 seconds

Estimate Number	C-1 $f_{ahp}=1000$ $f_{bhp}=2000$	C-2 $f_{ahp}=2000$ $f_{bhp}=4000$	C-3 $f_{ahp}=4000$ $f_{bhp}=7000$	C-4 $f_{ahp}=7000$ $f_{bhp}=12000$	Estimate Number	C-1 $f_{ahp}=1000$ $f_{bhp}=2000$	C-2 $f_{ahp}=2000$ $f_{bhp}=4000$	C-3 $f_{ahp}=4000$ $f_{bhp}=7000$	C-4 $f_{ahp}=7000$ $f_{bhp}=12000$
1	0.425	0.388	0.418	0.378	31	0.390	0.420	0.387	0.408
2	0.365	0.415	0.398	0.409	32	0.441	0.395	0.407	0.427
3	0.372	0.442	0.400	0.400	33	0.370	0.380	0.389	0.382
4	0.423	0.410	0.401	0.385	34	0.395	0.372	0.408	0.416
5	0.404	0.423	0.365	0.372	35	0.400	0.415	0.439	0.361
6	0.380	0.429	0.386	0.390	36	0.410	0.400	0.415	0.380
7	0.418	0.416	0.411	0.398	37	0.467	0.411	0.386	0.399
8	0.382	0.420	0.389	0.380	38	0.408	0.398	0.383	0.402
9	0.360	0.390	0.378	0.399	39	0.402	0.440	0.382	0.380
10	0.400	0.368	0.392	0.373	40	0.361	0.385	0.395	0.398
11	0.414	0.360	0.390	0.408	41	0.348	0.378	0.432	0.398
12	0.410	0.379	0.399	0.410	42	0.402	0.385	0.411	0.382
13	0.365	0.403	0.400	0.392	43	0.404	0.422	0.420	0.403
14	0.400	0.377	0.363	0.409	44	0.413	0.423	0.399	0.405
15	0.340	0.375	0.416	0.410	45	0.400	0.382	0.382	0.395
16	0.342	0.421	0.407	0.399	46	0.410	0.390	0.388	0.400
17	0.376	0.412	0.396	0.390	47	0.431	0.400	0.428	0.392
18	0.361	0.430	0.442	0.380	48	0.385	0.395	0.397	0.385
19	0.401	0.367	0.404	0.410	49	0.435	0.416	0.384	0.370
20	0.387	0.419	0.421	0.414	50	0.382	0.325	0.381	0.383
21	0.422	0.410	0.410	0.402	51	0.363	0.407	0.393	0.412
22	0.400	0.435	0.380	0.402	52	0.400	0.390	0.407	0.401
23	0.422	0.400	0.394	0.393	53	0.381	0.398	0.415	0.400
24	0.412	0.403	0.455	0.416	54	0.430	0.417	0.412	0.401
25	0.403	0.391	0.384	0.418	55	0.472	0.379	0.419	0.395
26	0.360	0.410	0.399	0.394	56	0.465	0.419	0.422	0.400
27	0.400	0.435	0.403	0.381	57	0.353	0.388	0.428	0.390
28	0.372	0.401	0.400	0.401	58	0.450	0.410	0.402	0.393
29	0.360	0.402	0.397	0.402	59	0.432	0.407	0.410	0.411
30	0.418	0.450	0.402	0.410	60	0.376	0.398	0.418	0.401
					61	0.400	0.412	0.402	0.378

Table 14.4 Probability Density Estimates for Different Frequency Ranges

Probability Density Estimates p_v for Center Frequencies of f_c cps.

rms level = 1 volt; Δx = 0.100 volts; v = 0 volts; K = 0.515 seconds; B = $(f_b - f_a)$ = 56 cps

Estimate No.	D-1 f_c = 100	D-2 f_c = 500	D-3 f_c = 1000	D-4 f_c = 4000	D-5 f_c = 7000	Estimate No.	D-1 f_c = 100	D-2 f_c = 500	D-3 f_c = 1000	D-4 f_c = 4000	D-5 f_c = 7000
1	0.480	0.380	0.430	0.343	0.437	31	0.461	0.446	0.554	0.418	0.345
2	0.440	0.411	0.454	0.432	0.473	32	0.404	0.460	0.374	0.434	0.617
3	0.368	0.393	0.368	0.420	0.468	33	0.352	0.360	0.360	0.416	0.379
4	0.339	0.399	0.446	0.400	0.372	34	0.480	0.434	0.468	0.420	0.459
5	0.420	0.423	0.344	0.381	0.344	35	0.429	0.432	0.388	0.442	0.361
6	0.320	0.440	0.372	0.472	0.501	36	0.336	0.428	0.350	0.460	0.353
7	0.369	0.352	0.422	0.371	0.371	37	0.400	0.370	0.380	0.442	0.454
8	0.442	0.488	0.386	0.365	0.403	38	0.463	0.371	0.376	0.410	0.430
9	0.400	0.402	0.416	0.350	0.421	39	0.379	0.427	0.410	0.369	0.496
10	0.400	0.404	0.350	0.421	0.407	40	0.354	0.400	0.404	0.364	0.372
11	0.370	0.455	0.404	0.368	0.439	41	0.410	0.510	0.372	0.454	0.399
12	0.340	0.396	0.428	0.403	0.354	42	0.348	0.409	0.430	0.419	0.413
13	0.392	0.408	0.406	0.382	0.370	43	0.310	0.347	0.508	0.407	0.412
14	0.358	0.480	0.398	0.418	0.412	44	0.380	0.364	0.360	0.492	0.460
15	0.329	0.441	0.388	0.410	0.334	45	0.380	0.440	0.404	0.428	0.395
16	0.411	0.404	0.454	0.446	0.422	46	0.513	0.443	0.380	0.412	0.411
17	0.338	0.480	0.418	0.370	0.439	47	0.390	0.362	0.364	0.405	0.392
18	0.384	0.427	0.452	0.421	0.381	48	0.477	0.388	0.406	0.440	0.359
19	0.356	0.436	0.384	0.391	0.405	49	0.392	0.438	0.436	0.462	0.370
20	0.394	0.451	0.394	0.353	0.375	50	0.306	0.397	0.400	0.390	0.397
21	0.321	0.420	0.364	0.462	0.360	51	0.322	0.395	0.468	0.381	0 396
22	0.403	0.432	0.372	0.361	0.380	52	0.360	0.401	0.380	0.401	0,402
23	0.425	0.406	0.330	0.410	0.400	53	0.300	0.364	0.426	0.418	0.430
24	0.350	0.360	0.436	0.428	0.408	54	0.352	0.406	0.306	0.465	0.403
25	0.391	0.409	0.422	0.480	0.368	55	0.389	0.397	0.428	0.420	0.410
26	0.360	0.410	0.384	0.392	0.365	56	0.366	0.497	0.368	0.343	0.401
27	0.400	0.482	0.448	0.410	0.487	57	0.419	0.373	0.404	0.420	0.359
28	0.374	0.403	0.496	0.390	0.332	58	0.482	0.382	0.410	0.421	0.302
29	0.395	0.389	0.418	0.430	0.461	59	0.431	0.360	0.402	0.434	0.382
30	0.388	0.382	0.362	0.363	0.499	60	0.347	0.412	0.380	0.385	0.452
						61	0.326	0.431	0.374	0.392	0.344

Table 14.5 Probability Density Estimates for Different Center Frequencies

Probability Density Estimates $\hat{p}_v$									
rms level = 1.00 volt; Δx = 0.100 volts; v = 0 volts; K = 0.109 seconds; f_{ahp} = 100 cps ; f_{bhp} = 600 cps									
Estimate Number	$\hat{p}_v$	Estimate Number	$\hat{p}_v$	Estimate Number	$\hat{p}_v$	Estimate Number	$\hat{p}_v$	Estimate Number	$\hat{p}_v$
1	0.385	38	0.405	75	0.357	112	0.378	149	0.409
2	0.390	39	0.362	76	0.369	113	0.402	150	0.410
3	0.358	40	0.342	77	0.367	114	0.370	151	0.350
4	0.418	41	0.380	78	0.441	115	0.420	152	0.403
5	0.379	42	0.370	79	0.517	116	0.419	153	0.500
6	0.344	43	0.430	80	0.445	117	0.440	154	0.402
7	0.365	44	0.290	81	0.303	118	0.391	155	0.413
8	0.368	45	0.448	82	0.368	119	0.315	156	0.434
9	0.360	46	0.419	83	0.410	120	0.418	157	0.430
10	0.348	47	0.378	84	0.427	121	0.424	158	0.315
11	0.418	48	0.363	85	0.481	122	0.417	159	0.355
12	0.437	49	0.402	86	0.375	123	0.453	160	0.361
13	0.442	50	0.402	87	0.412	124	0.432	161	0.451
14	0.378	51	0.439	88	0.372	125	0.425	162	0.392
15	0.425	52	0.379	89	0.436	126	0.380	163	0.362
16	0.422	53	0.377	90	0.439	127	0.391	164	0.344
17	0.420	54	0.377	91	0.390	128	0.355	165	0.420
18	0.357	55	0.360	92	0.418	129	0.378	166	0.482
19	0.401	56	0.408	93	0.409	130	0.416	167	0.376
20	0.551	57	0.330	94	0.410	131	0.470	168	0.423
21	0.362	58	0.456	95	0.391	132	0.351	169	0.432
22	0.470	59	0.420	96	0.380	133	0.352	170	0.355
23	0.400	60	0.365	97	0.390	134	0.428	171	0.393
24	0.348	61	0.360	98	0.415	135	0.415	172	0.380
25	0.452	62	0.417	99	0.410	136	0.401	173	0.480
26	0.365	63	0.435	100	0.409	137	0.363	174	0.430
27	0.343	64	0.441	101	0.362	138	0.393	175	0.422
28	0.390	65	0.425	102	0.408	139	0.369	176	0.470
29	0.391	66	0.376	103	0.365	140	0.420	177	0.438
30	0.385	67	0.459	104	0.380	141	0.363	178	0.381
31	0.485	68	0.420	105	0.424	142	0.428	179	0.377
32	0.422	69	0.452	106	0.377	143	0.374	180	0.480
33	0.340	70	0.398	107	0.445	144	0.441	181	0.482
34	0.334	71	0.376	108	0.496	145	0.394	182	0.410
35	0.401	72	0.397	109	0.386	146	0.450	183	0.435
36	0.434	73	0.310	110	0.367	147	0.352		
37	0.435	74	0.382	111	0.420	148	0.461		

Table 14.6 Probability Density Estimates for Normality Test

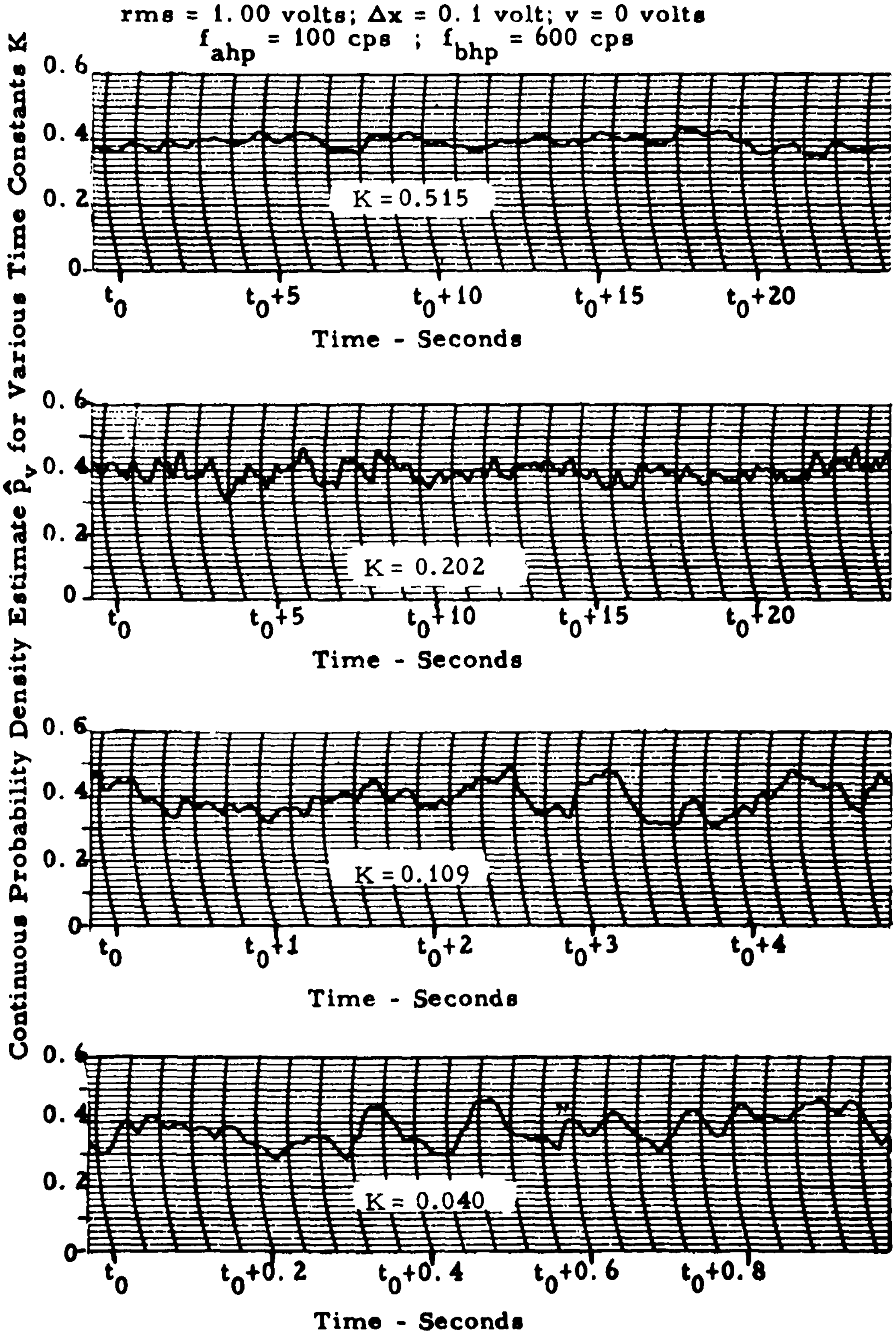

Figure 14.6 Examples of Actual Probability Density Data

Effective Frequency Bandwidths B and Expected Zero Crossings per Second $\overline{\nu}_0$ For Given Half Power Point Frequencies, f_{ahp} and f_{bhp}, or Given Center Frequencies, f_c							
Description of Experiment (Test Procedure)	f_{ahp} (cps)	f_{bhp} (cps)	f_a (cps)	f_b (cps)	f_c (cps)	B (cps)	$\overline{\nu}_0$ (crossings per second)
Procedures A and B (Tables 14.2 and 14.3)	100	600	95	630	--	535	788
Procedure C-1 (Table 14.4)	1000	2000	950	2100	--	1150	3120
Procedure C-2 (Table 14.4)	2000	4000	1900	4200	--	2300	6240
Procedure C-3 (Table 14.4)	4000	7000	3800	7350	--	3550	11300
Procedure C-4 (Table 14.4)	7000	12000	6650	12600	--	5950	19600
Procedure D-1 (Table 14.5)	--	--	72	128	100	56	203
Procedure D-2 (Table 14.5)	--	--	472	528	500	56	1000
Procedure D-3 (Table 14.5)	--	--	972	1028	1000	56	2000
Procedure D-4 (Table 14.5)	--	--	3972	4028	4000	56	8000
Procedure D-5 (Table 14.5)	--	--	6972	7028	7000	56	14000

Table 14.7 Effective Bandwidths and Expected Zero Crossings

Sample Mean $\bar{p}$ and Sample Variance s^2 for Probability Density Estimates $\hat{p}_v$			
Description of Experiment (Test Procedures)	Sample Size N	Sample Mean $\bar{p}$	Sample Variance s^2
Procedures A (Table 14.2)			
A-1	61	0.3939	0.000318
A-2	↓	0.3928	0.000665
A-3		0.3932	0.00187
A-4		0.4020	0.00537
Procedures B (Table 14.3)			
B-1		0.3356	0.00116
B-2		0.2300	0.000764
B-3		0.1201	0.000403
B-4		0.0463	0.000217
B-5		0.0141	0.0000778
Procedures C (Table 14.4)			
C-1		0.3979	0.000897
C-2		0.4023	0.000490
C-3		0.4023	0.000311
C-4		0.3963	0.000179
Procedures D (Table 14.5)			
D-1		0.3865	0.00259
D-2		0.4132	0.00139
D-3		0.4035	0.00185
D-4		0.4100	0.00119
D-5		0.4056	0.00268

Table 14.8 Sample Means and Variances for Probability Density Estimates

14.4.4 Experimental Values for the Coefficient and Test for Equivalence

The experimentally determined values for the coefficient C_0 in Eq. (14.15) are presented in Table 14.9. An estimate $\hat{C}_0$ is computed for each experiment assuming the parameter $X = B$ and $X = \overline{\nu}_0$.

(a) Equivalence of Coefficients Based on Bandwidth

From Table 14.9, the values for $\hat{C}_0$ based on the bandwidth B are now tested for equivalence by the procedures detailed in Section 14.2.3(a). Using Eqs. (14.36) and (14.37)

$$s^2_{\hat{C}_0} = 0.000203 \qquad \sigma^2_{\hat{C}_0} = 0.0000426$$

The hypothesis H_0 in Eq. (14.38) is tested using Eq. (14.40) as follows.

$$\frac{s^2_{\hat{C}_0}}{\sigma^2_{\hat{C}_0}} = 4.76 \nleq 1.53$$

Thus, the hypothesis of equivalence is rejected at the $\alpha = 0.05$ level of significance, and there is reason to believe that the uncertainty expression given by Eq. (14.15) with $X = B$ is not valid.

(b) Equivalence of Coefficients Based on Zero Crossings

Again referring to Table 14.9, consider now the values for $\hat{C}_0$ based upon the zero crossings $\overline{\nu}_0$. By observation, it is obvious that the values for $\hat{C}_0$ are not equivalent for those cases where the bandwidth is narrow as represented by the data for Test Procedure D. However, consider the values for $\hat{C}_0$ excluding Test Procedure D; that is, the values for $\hat{C}_0$ determined by Test Procedures A, B, and C only. These 13 values are now tested for equivalence by the procedures detailed in Section 14.2.3(a).

For the case of $M = 13$, the region of acceptance for the hypothesis H_0 is as follows.

$$\frac{s^2_{\hat{C}_0}}{\sigma^2_{\hat{C}_0}} \leq 1.62 \qquad (14.48)$$

Experimental Values for the Coefficient C_0 Based Upon the Number of Zero Crossings per Second $\bar{\nu}_0$, and Based Upon the Frequency Bandwidth B								
Test Procedure	Δx (volts)	$T_1 = 2K$ (seconds)	B (Table 14.7) (cps)	$\bar{\nu}_0$ (Table 14.7) $\left(\frac{\text{crossings}}{\text{second}}\right)$	$\bar{p}$ (Table 14.8)	s^2 (Table 14.8)	$\hat{C}_0$ based on B $\left[\frac{s^2 \Delta x T_1 B}{\bar{p}}\right]$	$\hat{C}_0$ based on $\bar{\nu}_0$ $\left[\frac{s^2 \Delta x T_1 \bar{\nu}_0}{\bar{p}}\right]$
A-1	0.100	1.03	535	788	0.3939	0.000318	0.0445	0.0655
A-2		0.404	535	788	0.3928	0.000665	0.0366	0.0540
A-3		0.218	535	788	0.3932	0.00187	0.0554	0.0819
A-4		0.080	535	788	0.4020	0.00537	0.0572	0.0841
B-1		0.218	535	788	0.3356	0.00116	0.0402	0.0596
B-2		0.218	535	788	0.2300	0.000764	0.0387	0.0570
B-3		0.218	535	788	0.1201	0.000403	0.0391	0.0577
B-4		0.218	535	788	0.0463	0.000217	0.0546	0.0806
B-5		0.218	535	788	0.0141	0.0000778	0.0643	0.0949
C-1		0.080	1150	3120	0.3979	0.000897	0.0207	0.0563
C-2		0.080	2300	6240	0.4023	0.000490	0.0224	0.0606
C-3		0.080	3550	11300	0.4023	0.000311	0.0219	0.0698
C-4		0.080	5950	19600	0.3963	0.000179	0.0215	0.0709
D-1		1.03	56	203	0.3865	0.00259	0.038	0.14
D-2		1.03	56	1000	0.4132	0.00139	0.019	0.35
D-3		1.03	56	2000	0.4035	0.00185	0.026	0.94
D-4		1.03	56	8000	0.4100	0.00119	0.017	2.4
D-5		1.03	56	14000	0.4056	0.00268	0.038	9.5

Table 14.9 Experimental Values for the Coefficient C_0

Using Eqs. (14. 36) and (14. 37),

$$s^2_{\hat{C}_0} = 0.000156 \quad \text{and} \quad \sigma^2_{\hat{C}_0} = 0.000152$$

The hypothesis H_0 in Eq. (14. 38) is tested using Eq. (14. 48) as follows.

$$\frac{s^2_{\hat{C}_0}}{\sigma^2_{\hat{C}_0}} = 1.03 < 1.62$$

Thus, the hypothesis of equivalence is accepted at the $\alpha = 0.05$ level of significance, and the uncertainty expression given by Eq. (14. 15) for $X = \overline{\nu}_0$ is considered valid for the conditions of Test Procedures A, B, and C.

(c) Detailed Evaluation of Coefficients Based on Bandwidth

It is apparent from practical considerations that the values for $\hat{C}_0$ obtained under Test Procedure D are equivalent when based upon bandwidth. The only parameter being changed in Procedure D is the center frequency of a constant bandwidth filter which changes only the expected number of zero crossings. It is clear from observation that the values for $\hat{C}_0$ based on zero crossings are greatly affected while the values based on bandwidth show only scatter with no distinct trend. The scatter is large because of practical difficulties encountered in the experiments with narrow bandwidth signals. The sampling errors for the data gathered by Test Procedure D are undoubtedly greater than theoretically predicted by Eq. (14. 34). It is then desirable to test the equivalence of the values for $\hat{C}_0$ based on bandwidth for Test Procedures A, B, and C alone, as was done in (b) for the data based on zero crossings.

For the case of M = 13, the region of acceptance for the hypothe- of equivalence, H_0, is as given in Eq. (14. 48). Using Eqs. (14. 36) and (14. 37),

$$s^2_{\hat{C}_0} = 0.000209 \quad \text{and} \quad \sigma^2_{\hat{C}_0} = 0.0000509$$

The hypothesis H_0 in Eq. (14. 38) is tested using Eq. (14. 48) as follows.

$$\frac{s^2_{\hat{C}_0}}{\sigma^2_{\hat{C}_0}} = 4.11 \neq 1.62$$

Thus, the hypothesis of equivalence based on $X = B$ is rejected at the $\alpha = 0.05$ level of significance. This result augments the conclusion arrived at in (b) of this section; namely, the uncertainty expression given by Eq. (14. 15) for $X = \overline{\nu}_0$ is appropriate for the conditions of Test Procedures A, B, and C.

In summary, it appears that Eq. (14. 15) with $X = \overline{\nu}_0$ is a valid expression for the normalized variance of probability density estimates as long as the signal bandwidth B is relatively broad. On the other hand, for narrow bandwidths, Eq. (14. 15) with $X = B$ appears to be appropriate. These matters are considered in more detail in Section 14. 5.

14. 4. 5 Test of Normality for Probability Density Estimates

The 183 probability density estimates presented in Table 14. 6 are now tested for normality by a "chi-squared goodness of fit test" using 16 class intervals. A histogram for the data is shown in Figure 14. 7. The computation of χ^2 is presented in Table 14. 10. Note that each estimate is based upon a number of events $n \approx 9$ using Eq. (14. 8) or $n \approx 17$ using Eq. (14. 11).

Let it be hypothesized that the estimates $\hat{p}_v$ are normally distributed. From Eq. (14. 43), this hypothesis will be accepted at the $\alpha = 5\%$ level of significance if

$$\chi^2 < 22.36$$

From Table 14. 10,

$$\chi^2 = 19.85 < 22.36$$

so the hypothesis is accepted. On statistical grounds, it is concluded that there is no reason to suspect that differences between the distribution of $\hat{p}_v$ and a normal distribution are significant.

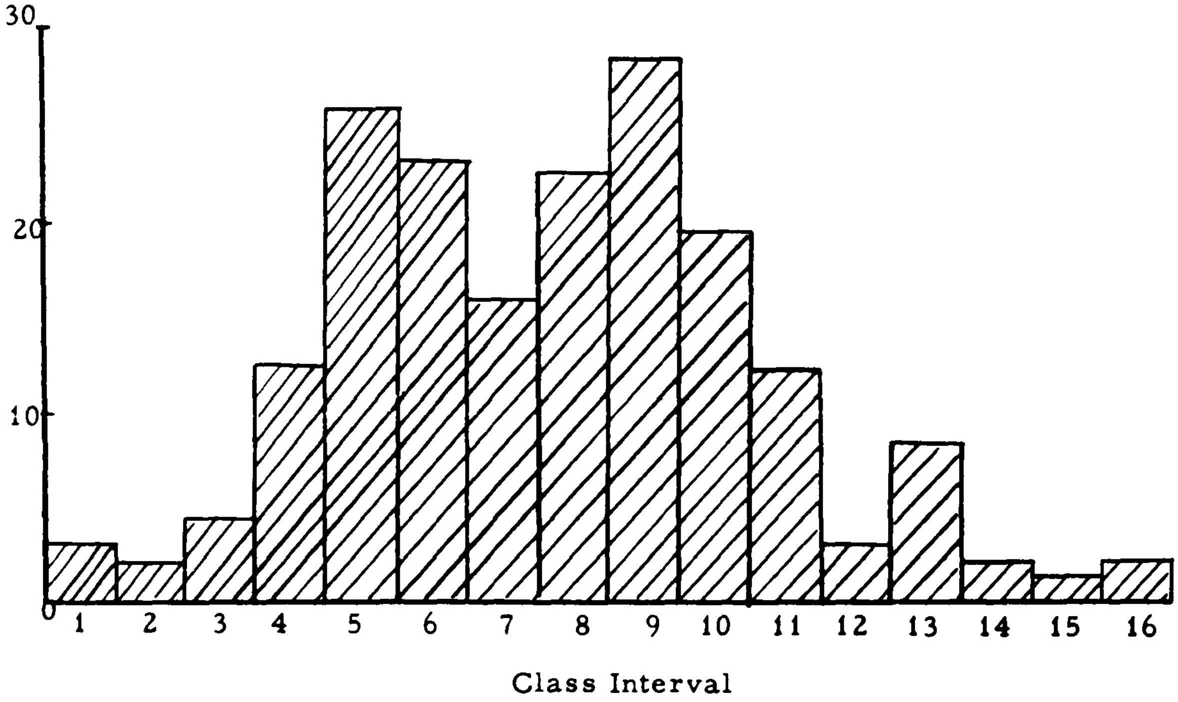

Figure 14.7 Frequency Histogram for Probability Density Estimates

Class Number	Interval		Midpoint	f	* $x_c = \frac{x - 0.405}{0.014}$	$f x_c$	$f x_c^2$	$z_i = \frac{x_c + 0.5 - \bar{x}_c}{s_c}$	ΔA	$F = N(\Delta A)$	$\lvert F - f \rvert$	$\frac{(F-f)^2}{F}$
	x_i	x_{i+1}	x									
1	-∞	0.314	0.307	3	-7	-21		-2.198	0.014	2.6	0.4	0.06
2	0.314	0.328	0.321	2	-6	-12		-1.851	0.018	3.3	1.3	0.51
3	0.328	0.342	0.335	4	-5	-20		-1.504	0.035	6.4	2.4	0.90
4	0.342	0.356	0.349	12	-4	-48		-1.156	0.056	10.2	1.8	0.31
5	0.356	0.370	0.363	26	-3	-78		-0.809	0.086	15.7	10.3	6.75
6	0.370	0.384	0.377	23	-2	-26		-0.462	0.114	20.9	2.1	0.21
7	0.384	0.398	0.391	16	-1	-16		-0.114	0.133	24.3	8.3	2.83
8	0.398	0.412	0.405	22	0	0		0.233	0.135	24.7	2.7	0.29
9	0.412	0.426	0.419	28	1	28		0.581	0.128	23.4	4.6	0.90
10	0.426	0.440	0.433	19	2	38		0.928	0.105	19.2	0.2	0.00
11	0.440	0.454	0.447	12	3	36		1.275	0.076	13.9	1.9	0.25
12	0.454	0.468	0.461	3	4	12		1.623	0.047	8.6	5.6	3.64
13	0.468	0.482	0.475	8	5	40		1.970	0.029	5.3	2.7	1.37
14	0.482	0.496	0.489	2	6	12		2.318	0.014	2.6	0.6	0.13
15	0.496	0.510	0.503	1	7	7		2.665	0.006	1.1	0.1	0.01
16	0.510	∞	0.517	2	8	16		∞	0.004	0.7	1.3	1.69
						-32	1522					19.85

*A linear transformation is applied at this point (transform to "coded" data) to simplify the numerical computations.

$N = \sum f = 183$ d.f. $= k - 3 = 13$ $\bar{x}_c = \left(\sum f x_c\right)/N = -0.17$ $s_c^2 = \left[\left(\sum f x_c^2\right)/N\right] - \bar{x}_c^2 = 8.288$

Table 14.10 Computations for Chi-Squared Goodness of Fit Test

14.5 DISCUSSION OF RESULTS

14.5.1 Evaluation of Estimation Uncertainty

From the results presented in Section 14.4.4, it appears that the normalized variance for probability density estimates is not a simple function of only zero crossings or frequency bandwidth for all cases. For those cases where the signal bandwidth is relatively wide, Eq. (14.15) with $X = \overline{\nu}_0$ seems to be the best mathematical model for the normalized variance of probability density estimates. By averaging the 13 values for $\hat{C}_0$ under Test Procedures A, B, and C in Table 14.9, the normalized variance and standard error for this case are

$$\epsilon^2 = \frac{0.068}{\Delta x\, \hat{p}(x)\, \overline{\nu}_0\, T_1} \tag{14.49a}$$

$$\epsilon = \frac{0.26}{\sqrt{\Delta x\, \hat{p}(x)\, \overline{\nu}_0\, T_1}} \tag{14.49b}$$

On the other hand, for those cases where the signal bandwidth is relatively narrow, Eq. (14.15) with X = B is the best model for the normalized variance of probability density estimates. By averaging the five values for $\hat{C}_0$ under Test Procedure D in Table 14.9, the normalized variance and standard error for this case are

$$\epsilon^2 = \frac{0.028}{\Delta x\, \hat{p}(x)\, B T_1} \tag{14.50a}$$

$$\epsilon = \frac{0.17}{\sqrt{\Delta x\, \hat{p}(x)\, B T_1}} \tag{14.50b}$$

The question that arises is when does Eq. (14.50) apply instead of Eq. (14.49). The data gathered is not sufficient to firmly answer this question. However, based upon the data available, it appears that the critical relationship is the ratio of bandwidth to expected zero crossings. When this ratio is very small, as occurs for narrow bandwidths on high center frequencies, Eq. (14.50) applies. For the case where the bandwidth to zero crossing ratio is, say, greater than one-third, Eq. (14.49) applies. Thus, it is believed that Eq. (14.49) applies to low frequency

signals where the bandwidth is narrow but the expected number of zero crossings is also small.

It appears that the theoretical developments for the estimation uncertainty in Section 14. 1. 2 are not yet in final complete form. The experimental results in Section 14. 4. 4 tend to indicate the true normalized variance for probability density estimates is a complicated function of both the expected number of zero crossings and the frequency bandwidth of the signal being investigated. However, it is also possible that the specific probability density analyzer employed for these experiments does not functionally produce estimates in rigorous compliance with the mathematical models developed in Section 14. 1. 1. These possibilities are left unresolved.

It is interesting to consider possible mathematical formulas for the uncertainty of probability density estimates which produce a reasonable fit to the data gathered in these experiments. By inspection and trial and error, the following equation for the normalized variance of estimates is presented without theoretical justification. Other equations might be equally applicable. Assume

$$\epsilon^2 = \frac{C_0}{\Delta x\, \hat{p}(x)\, B\left[1 - (B/\overline{\nu}_0)^2\right] T_1} \tag{14.51}$$

Using the data presented in Table 14. 9, the values for $\hat{C}_0$ resulting from the formula presented in Eq. (14. 51) are computed as shown in Table 4. 11.

From the data in Table 14. 11, it is seen that the resulting values for $\hat{C}_0$ are, in qualitative terms, reasonably equivalent. By averaging the 18 values for $\hat{C}_0$ in Table 14. 11, the normalized variance and standard error for this case are

$$\epsilon^2 = \frac{0.025}{\Delta x\, \hat{p}(x)\, B\left[1 - (B/\overline{\nu}_0)^2\right] T_1} \tag{14.51a}$$

$$\epsilon = \frac{0.16}{\sqrt{\Delta x\, \hat{p}(x)\, B\left[1 - (B/\overline{\nu}_0)^2\right] T_1}} \tag{14.51b}$$

Experimental Values for the Coefficient C_0 Based Upon the Assumed Formula of Eq. (14.51)				
Test Procedure	$\hat{C}_0$ based on B Table 14.9	$(B/\bar{V}_0)^2$	$1-(B/\bar{V}_0)^2$	$\hat{C}_0$ based on $B[1-(B/\bar{V}_0)^2]$
A-1	0.0445	0.461	0.539	0.0240
A-2	0.0366	0.461	0.539	0.0197
A-3	0.0554	0.461	0.539	0.0299
A-4	0.0572	0.461	0.539	0.0308
B-1	0.0402	0.461	0.539	0.0217
B-2	0.0387	0.461	0.539	0.0208
B-3	0.0391	0.461	0.539	0.0211
B-4	0.0546	0.461	0.539	0.0294
B-5	0.0643	0.461	0.539	0.0346
C-1	0.0207	0.135	0.865	0.0179
C-2	0.0224	0.135	0.865	0.0194
C-3	0.0219	0.099	0.901	0.0197
C-4	0.0215	0.092	0.908	0.0195
D-1	0.038	0.003	0.997	0.038
D-2	0.019	0.001	0.999	0.019
D-3	0.026	0.000	1.000	0.026
D-4	0.017	0.000	1.000	0.017
D-5	0.038	0.000	1.000	0.038

Table 14.11 Experimental Value for the Coefficient C_0

Although Eq. (14. 51) presents a good fit to the experimental data gathered herein, there is no justification for its general use since the equation was arrived at by arbitrary methods without a firm theoretical foundation. Nevertheless, Eq. (14. 51) presents an interesting consideration for future theoretical and experimental studies of the uncertainty of probability density estimates.

14. 5. 2 Estimation Uncertainty and Measurement Parameters for Different Signals

Consider the hypothetical case where the probability density function for a stationary random signal is to be estimated so that the normalized standard error is no greater than 10%, i. e., $\epsilon = 0.10$. Assume that the low frequency cutoff of the signal is near DC so that Eq. (14. 49) applies. Also assume the analysis is to be performed with an amplitude window width of 0. 1 for a signal rms level of unity. For the case of a Gaussian probability density function, the association among the values for $\hat{p}(x)$, $\bar{\nu}_0$, and T_1 is as shown in Table 14. 12.

$\bar{\nu}_0$	Required Record Length $T > T_1$ seconds for Various Amplitude Levels v. $\epsilon = 0.10$ $\Delta x = 0.1$ rms = 1.0			
	$v = 1.0$ $\hat{p}_v = 0.2420$	$v = 2.0$ $\hat{p}_v = 0.0540$	$v = 3.0$ $\hat{p}_v = 0.0044$	$v = 4.0$ $\hat{p}_v = 0.0001$
10	28.	126.	1540.	68000
100	2. 8	12. 6	154.	6800
1000	0. 28	1. 26	15. 4	680
10000	0. 028	0. 126	1. 54	68

Table 14. 12 Record Length Required for 10% Standard Error

The case of $\bar{\nu}_0 = 10$ represents a signal which is very low frequency in nature; for example, the normal mode response of a flight vehicle to gust loads. From Table 14. 12, a record length of 1540 seconds (25.7

minutes) would be required to obtain estimates out to $\pm$ 3 rms with a normalized standard error of 10%. A record length of almost 19 hours would be required to obtain estimates out to $\pm$ 4 rms.

The case of $\bar{\nu}_0$ = 1000 represents a signal with a frequency range characteristic of the vibration response of local structure in flight vehicles. From Table 14.12, a record length of about 16 seconds would be required to obtain estimates with ϵ = 0.10 out to $\pm$ 3 rms, or about 11 minutes out to $\pm$ 4 rms.

The case of $\bar{\nu}_0$ = 10000 represents a signal with a frequency range typical of acoustic environments in flight vehicles. Here, a record length of only 1.6 seconds is required to obtain estimates with ϵ = 0.1 out to $\pm$ 3 rms, or a little over a minute out to $\pm$ 4 rms.

The above examples clearly illustrate the real problems in obtaining statistically accurate probability density estimates for signals which are low frequency in nature. For the extreme amplitudes in particular, relatively long sample records are required to obtain accurate probability density estimates, even for signals in the audio frequency range.

14.6 CONCLUSIONS

The uncertainty of probability density estimates has been investigated by carefully designed laboratory experiments. Two slightly different theoretical expressions for uncertainty are compared to experimental data. The results indicate that Eq. (14.49) is a valid representation for the normalized variance of probability density estimates for a wide range of practical conditions. However, for narrow bandwidth signals, a slightly different theoretical expression given by Eq. (14.50) appears to be a more valid representation for the normalized variance of estimates.

Neither of the two theoretical uncertainty expressions considered appears to be completely valid for all of the conditions studied. It is believed that the true expression for estimation uncertainty is more general and perhaps involves more terms than the expressions developed herein. Although these matters are not completely resolved, an empirical expression for the normalized variance of estimates is given by Eq. (14.51) which is a reasonably good fit for the data gathered. This empirical expression presents a guideline and motivation for future theoretical and experimental studies of the uncertainty of probability density estimates.

14.7 REFERENCES

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15. TESTS FOR RANDOMNESS

15.1 THEORY OF TESTS FOR RANDOMNESS

15.1.1 General Remarks

Before discussing quantitative procedures for testing vibration response signals for randomness, it is helpful to review certain qualitative relationships which distinguish random signals from periodic signals. Consider the properties of a vibration response which would be measured from sample records as a normal part of the data analyses. These properties would probably include a power spectrum (and/or an autocorrelation function), and an amplitude probability density function.

The most common property employed to describe a random vibration response is the power spectrum, G(f). A periodic vibration response would appear in a properly resolved power spectral density estimate as one or more sharp peaks. However, a sharp peak in the power spectrum may also represent the narrow band random response of a lightly damped structural resonance. A narrow band random response would always have a finite bandwidth while a sinusoidal response would theoretically appear as a delta function with no bandwidth. Unfortunately, the resolution of actual measurement procedures is usually not sufficient to distinguish between these two cases, as illustrated in Figure 15.1.

If an autocorrelation function is available, the problem of distinguishing between a sinusoidal vibration response and a narrow band random vibration response is greatly simplified. The autocorrelation function, $R(\tau)$, for a narrow band random vibration response will continually decay towards zero as the time displacement τ becomes large. However, if a sine wave is present, the autocorrelation function will approach a steady state oscillation as the time displacement becomes large, as illustrated in Figure 15.2.

Autocorrelation analysis presents a near foolproof method of detecting sinusoids in an otherwise random vibration response. However, autocorrelation functions are rarely determined when vibration data is analyzed using analog instruments. On the other hand, when vibration data is analyzed by digital techniques, autocorrelation functions are often computed as an intermediate step to obtain power spectra [G(f) is the Fourier transform of $R(\tau)$ for stationary random signals]. In this case, an autocorrelation function can be made available for detection of periodicities.

The presence of periodic components in a vibration response will also be revealed by the amplitude probability density function for the response. A random vibration response will, in most cases, have a probability density function which at least resembles the familiar bell-shaped Gaussian characteristic. This is particularly true for a narrow band random response (assuming the structural characteristics are reasonably linear), because linear narrow band filtering of random signals tends to suppress deviations from normality in the amplitude distribution characteristics. On the other hand, a sinusoidal vibration response will have a dish-shaped probability density function which is truncated at $\pm A = \sqrt{2}\, \sigma_x$, as illustrated in Figure 15. 3.

The above discussions illustrate how periodicities in an otherwise random vibration response may often be detected by simple observations from commonly analyzed data. However, these interpretations are fully effective only when sampling errors are small. If the available sample records (or analyses averaging times) are short, the uncertainty in the resulting measurements may mask the desired descriptive details in the various estimated properties. Furthermore, it is often desirable to determine if periodicities are present in an otherwise random vibration response before the data analysis has proceeded to the point of estimating correlation or probability density functions. A quantitative procedure for testing sample records for randomness by analysis of some elementary property of the vibration signal is clearly desirable.

In Ref. [1], Section 6. 1. 5, a test for randomness based upon an analysis of zero crossings is proposed. This procedure suggested in Ref. [1] will hereafter be referred to as Randomness Test A. An additional test for randomness based upon an analysis of mean square measurements is developed and presented herein. This second procedure will hereafter be referred to as Randomness Test B.

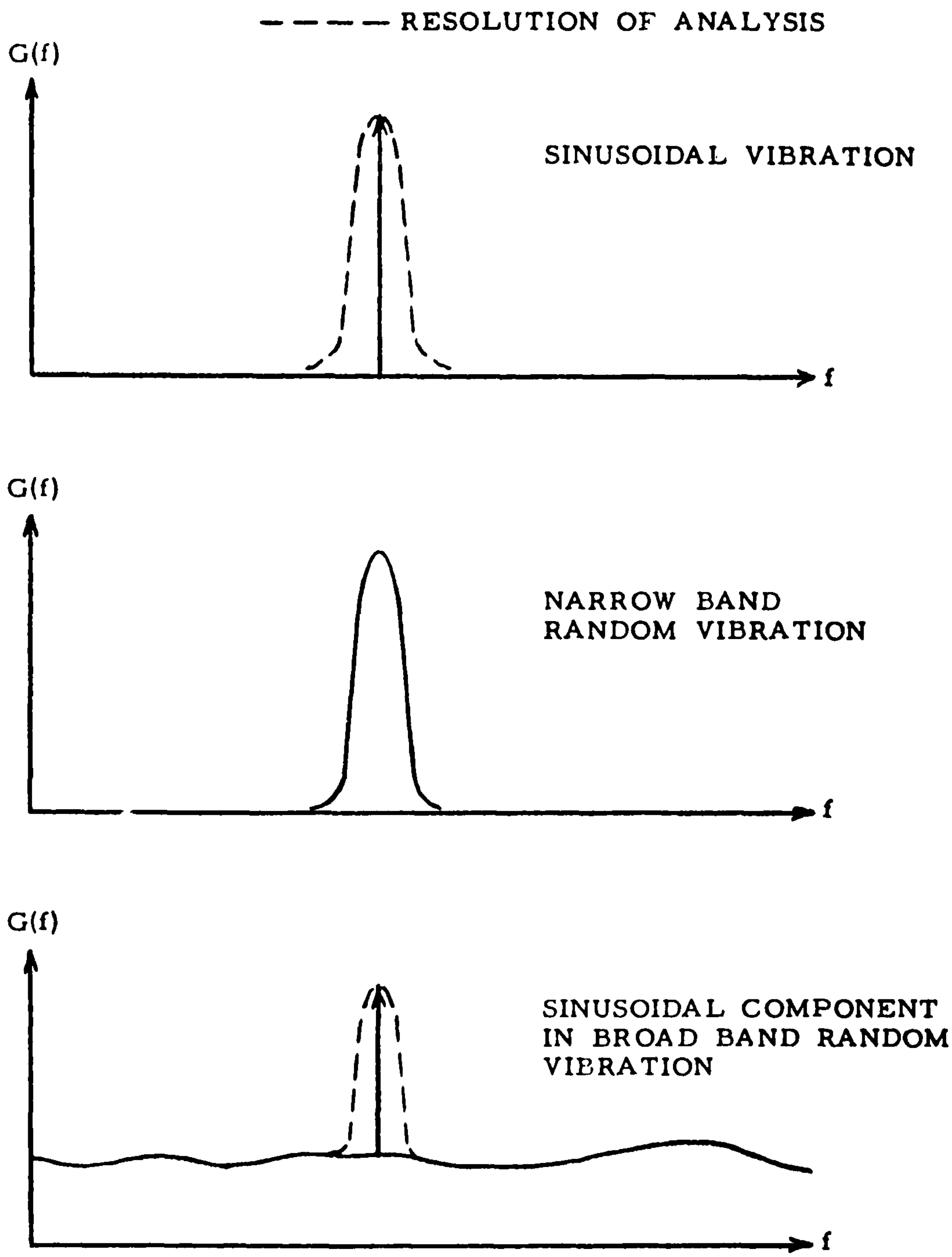


Figure 15.1 Power Spectral Density Functions

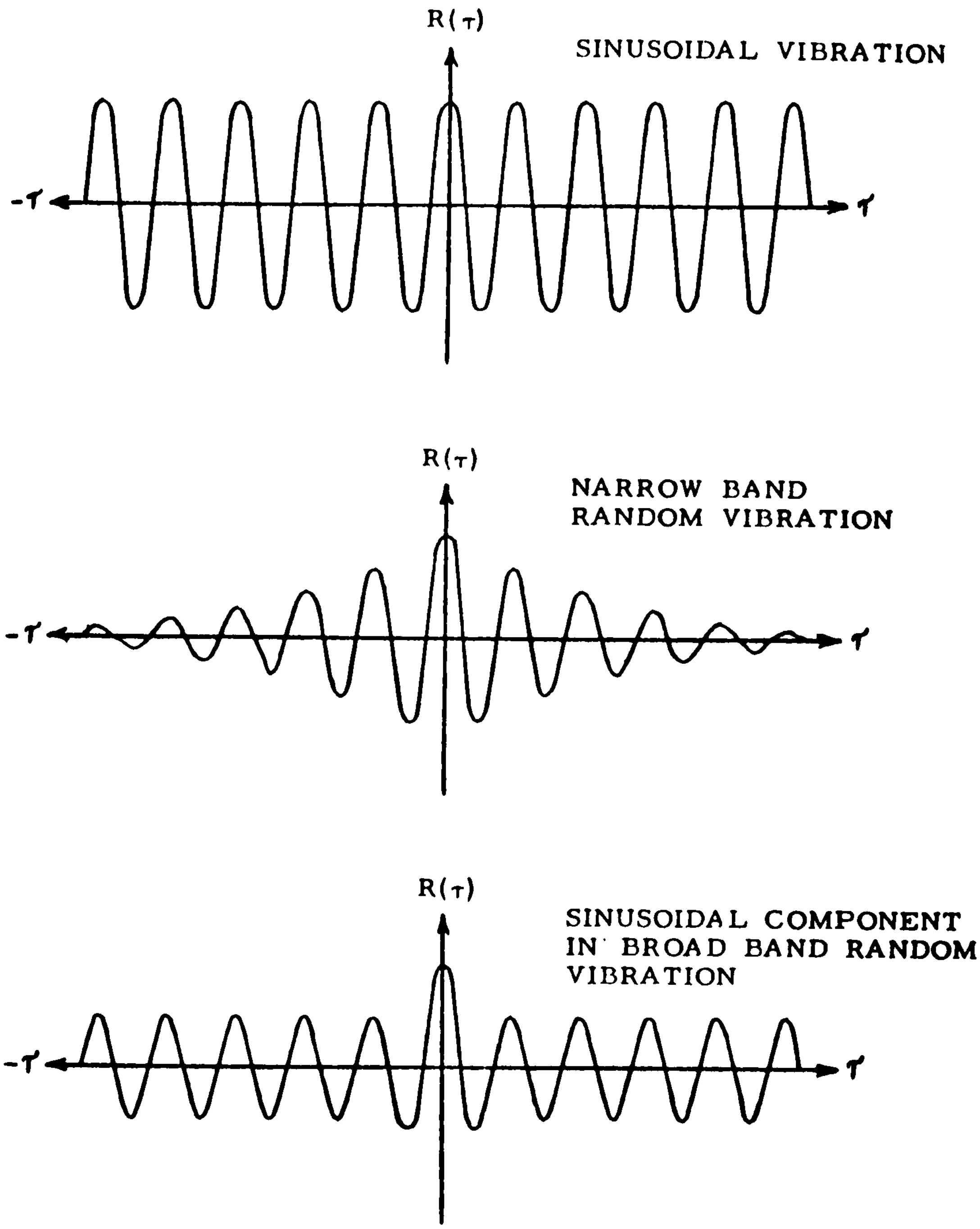


Figure 15.2 Autocorrelation Functions

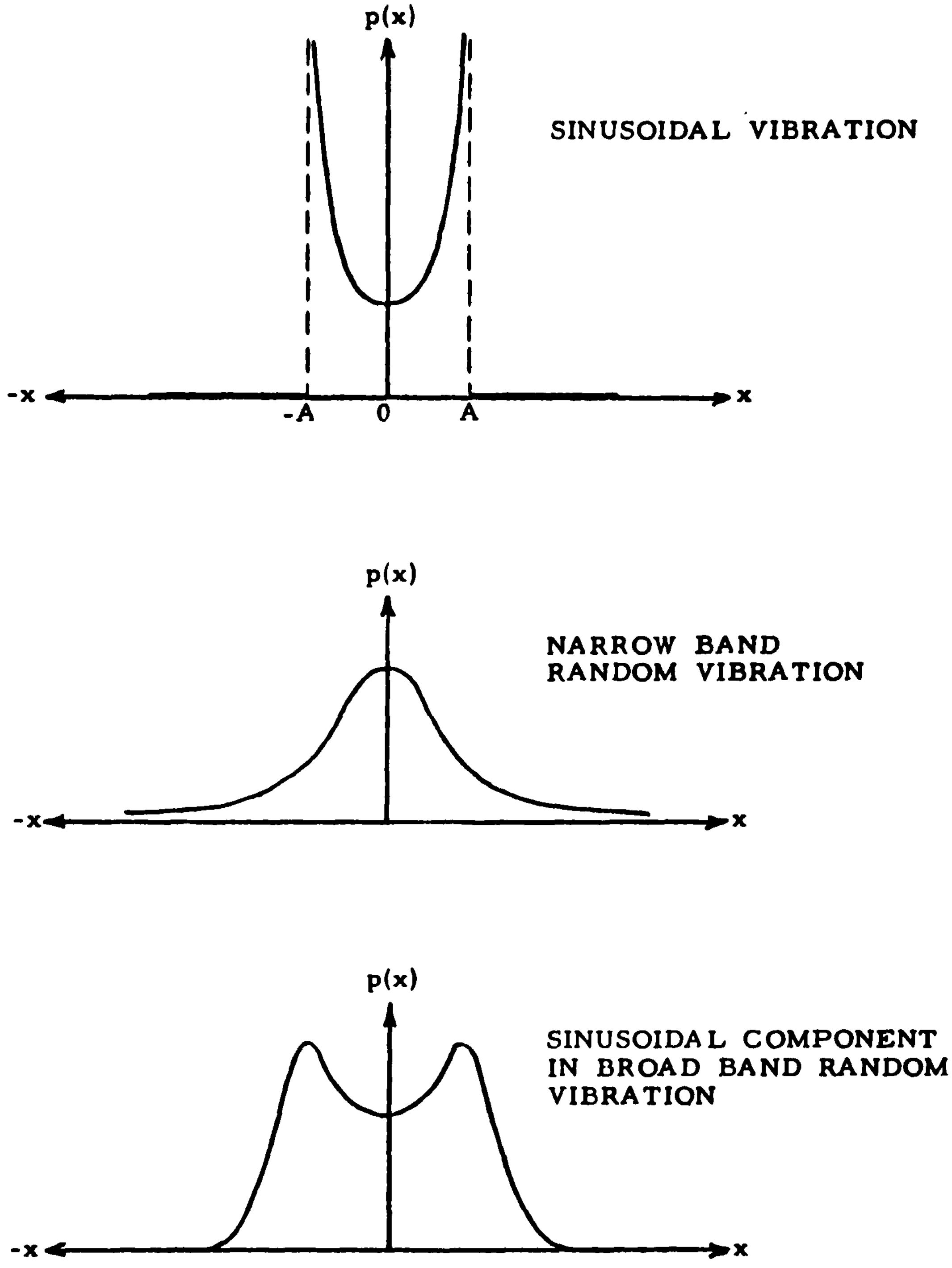


Figure 15.3 Amplitude Probability Density Functions

15.1.2 Review of Randomness Test A

Consider a stationary random signal x(t) with a mean value of zero. Assume x(t) has an approximately Gaussian probability density function and a uniform power spectrum between the frequency limits f_a and f_b.

For a sample record of length T, the equivalent number of events is given by $n = 2\overline{\nu}_0 T$ where $\overline{\nu}_0$ is the expected number of zero crossings per second. If n is large, say greater than 50, the number of zero crossings ν_0 measured from the sample record will have an approximately normal sampling distribution with a mean and variance as follows.

$$\mu_{\nu_0} = \frac{n}{2} \tag{15.1a}$$

$$\sigma^2_{\nu_0} = \frac{n(\frac{1}{2} n - 1)}{2(n - 1)} \tag{15.1b}$$

For a signal with a uniform power spectrum between the lower and upper frequency limits, f_a and f_b, respectively, the term $\overline{\nu}_0$ is given by

$$\overline{\nu}_0 = 2\left[\frac{f_a^2 + f_a f_b + f_b^2}{3}\right]^{1/2} \tag{15.2}$$

The practical validity of Eqs. (15.1) and (15.2) has been fully substantiated by the experiments presented in Section 12.

The above relationships may be employed as a test for randomness. Assume a sample record of length T is obtained from a stationary vibration response x(t). The number of zero crossings ν_0 in the sample record is counted. If x(t) is random, the measured number of zero crossings ν_0 should be equivalent to the expected number of zero crossings μ_{ν_0}, given theoretically by Eq. (15.1). Then a hypothesis of randomness is established by the null hypothesis H_0 as follows.

$$H_0 : \nu_0 = \mu_{\nu_0} \tag{15.3}$$

If H_0 is tested at the α level of significance, the region of acceptance is given by

$$-\sigma_{\nu_0} z_{\alpha/2} \leq (\nu_0 - \mu_{\nu_0}) \leq \sigma_{\nu_0} z_{\alpha/2} \tag{15.4}$$

where $z_{\alpha/2}$ is a normal deviate.

Example:

The above discussions will be clarified by considering an example. Assume that a three-second long sample record is obtained from a stationary vibration response signal. The vibration signal is to be tested for randomness at the 5% level of significance by investigating the zero crossings in the sample record. Say that the frequency range of interest is between 20 cps and 500 cps, and that the vibration signal has been filtered with ideally sharp cutoffs at these frequencies. Further, assume that the vibration signal, if random, has a uniform power spectrum and an approximately Gaussian probability density function.

Let it be hypothesized that the sample record was obtained from a random vibration signal. Then, from Eq. (15.2), the expected number of zero crossings per second will be

$$\overline{\nu}_0 = 2\left[\frac{(20)^2 + (20)(500) + (500)^2}{3}\right]^{1/2} = 589.2 \text{ crossings}$$

and the equivalent number of events represented by the sample record is

$$n = 2(589.2)3 = 3535$$

The value of n is sufficiently large to permit a normality assumption for the sampling distribution. From Eq. (15.1), the distribution for the actual number of zero crossings ν_0 in the sample record will have a mean value, variance, and standard deviation as follows.

$$\mu_{\nu_0} = \frac{1}{2}(3535) = 1768 \text{ crossings}$$

$$\sigma_{\nu_0}^2 = \frac{3535(\frac{1}{2}3535 - 1)}{2(3535 - 1)} = 883.5$$

$$\sigma_{\nu_0} = \sqrt{883.5} = 29.7 \text{ crossings}$$

To test at a 5% level of significance ($\alpha = 0.05$), the region of acceptance will be

$$\mu_{\nu_0} \pm 1.96\,\sigma_{\nu_0} = 1768 \pm 58 = (1710, 1826) \text{ crossings}$$

The actual number of zero crossings ν_0 which occurred in the three-second long sample record would now be counted. If ν_0 were between 1710 and 1826, the hypothesis of randomness would be accepted, i. e., the vibration signal would be considered as random. If ν_0 were less than 1710 or greater than 1826, the hypothesis of randomness would be rejected, i. e., the vibration signal would be considered as nonrandom. The probability of a Type I error (risk of rejecting the hypothesis when in fact it is true) associated with the decision is $\alpha = 0.05$.

One might be interested in associating a Type II error probability (risk of accepting the hypothesis when in fact it is not true) with the decision. The development of a totally meaningful Type II error is not really feasible, because the numerous possible conditions of nonrandomness cannot be explicitly defined. However, if one cares to make several arbitrary assumptions, a Type II error with limited meaning can be applied to the test. For example, let the extent of nonrandomness of a vibration signal be defined by the expected number of zero crossings per second. Further, assume that the variance of the sampling distribution will be the same as the variance of samples obtained from a random signal. A Type II error may now be established in terms of this definition of nonrandomness.

For the previous illustration, consider the detection of a condition of nonrandomness defined by an expected number of zero crossings at least 5% different from the expected zero crossings for a random signal. Then, the mean value and standard deviation of the sampling distribution for the nonrandom signal will be

$$\mu_{\nu_0}(\text{low}) = 1680$$

$$\mu_{\nu_0}(\text{high}) = 1856$$

$$\sigma_{\nu_0} = 29.7$$

The probability of a sample with the above expected values falling within the $\alpha = 0.05$ region of acceptance (1710 to 1826) for the test is 0.312. Thus, the Type II error probability is 31.2% for detecting a 5% difference in the expected number of zero crossings. By the same procedure, the Type II error probability is 10.5% for detecting a 6% difference in expected zero crossings, 2.6% for detecting a 7% difference in expected zero crossings, etc.

15.1.3 Principles of Randomness Test B

The second test for randomness to be discussed here is totally different in concept from Test A discussed in Section 15.1.2. Randomness Test B is based upon an investigation of mean square measurements rather than zero crossing measurements.

Consider a stationary random signal $x(t)$ with a mean value of zero and a mean square value of σ_x^2. Assume $x(t)$ has an approximately Gaussian probability density function and an equivalent ideal bandwidth of B. It is important to note that the power spectrum need not be uniform as long as an equivalent ideal bandwidth can be determined. From Section 13.5.2, the sampling distribution for a mean square value estimate, $\overline{x^2}$, will have a normalized variance given by

$$\epsilon^2 = \frac{\mathrm{Var}\,(\overline{x^2})}{(\sigma_x^2)^2} \approx \frac{1}{BT_1} \tag{15.5}$$

where T_1 is the averaging time.

Now consider the case where $x(t)$ is not random. Specifically, assume a periodic component is present in an otherwise random signal with an equivalent ideal bandwidth of B cps. That is

$$x(t) = p(t) + r(t) \tag{15.6}$$

where $p(t)$ is the periodic portion of $x(t)$, and $r(t)$ is the random portion of $x(t)$.

Let the moments of $x(t)$, $p(t)$, and $r(t)$ be denoted by $m_i(x)$, $m_i(p)$, and $m_i(r)$, respectively, where

$$
\begin{aligned}
m_1(x) &= \lim_{T\to\infty} \frac{1}{T}\int_0^T x(t)\,dt = \mu_x \\
m_2(x) &= \lim_{T\to\infty} \frac{1}{T}\int_0^T x^2(t)\,dt = \sigma_x^2 - \mu_x^2 \\
m_3(x) &= \lim_{T\to\infty} \frac{1}{T}\int_0^T x^3(t)\,dt \\
m_4(x) &= \lim_{T\to\infty} \frac{1}{T}\int_0^T x^4(t)\,dt
\end{aligned}
\tag{15.7}
$$

The terms $m_i(p)$ and $m_i(r)$ are defined in a similar manner.

Assuming that the mean values of both the periodic and random portions are zero, the first four moments for the periodic and random portions are as follows

$$
\begin{aligned}
m_1(p) &= 0 & m_1(r) &= 0 \\
m_2(p) &= \sigma^2(p) & m_2(r) &= \sigma^2(r) \\
m_3(p) &= 0 & m_3(r) &= 0 \\
m_4(p) &= \left[\sigma^2(p)\right]^2 & m_4(r) &= 3\left[\sigma^2(r)\right]^2 \quad *
\end{aligned}
\tag{15.8}
$$

The variance of a measured mean square value based upon $n = 2BT_1$ number of events is theoretically given by

$$
\mathrm{Var}\,(\overline{x^2}) = \frac{m_4(x) - m_2^2(x)}{2BT_1} \tag{15.9}
$$

*See Ref. [2], p. 289, for a derivation.

From Eq. (15. 7), the value of the fourth moment, $m_4(x)$, is given by

$$\begin{aligned} m_4(x) &= \lim_{T\to\infty} \int_0^T \left[p(t) + r(t)\right]^4 dt \\ &= \lim_{T\to\infty} \int_0^T \left[p^4(t) + 4p^3(t)r(t) + 6p^2(t)r^2(t) + 4p(t)r^3(t) + r^4(t)\right] dt \end{aligned} \tag{15. 10}$$

Assuming that p(t) and r(t) are independent, and noting that the first and third moments of both are zero, Eq. (15. 10) reduces to the following.

$$m_4(x) = m_4(p) + 6m_2(p)\, m_2(r) + m_4(r) \tag{15. 11}$$

Now the second moment, $m_2(x)$, is given by

$$\begin{aligned} m_2(x) &= \lim_{T\to\infty} \int_0^T \left[p(t) + r(t)\right]^2 dt \\ &= \lim_{T\to\infty} \int_0^T \left[p^2(t) + 2p(t)\, r(t) + r^2(t)\right] dt \\ &= m_2(p) + m_2(r) = \sigma_x^2 \qquad \text{since } \mu_x = 0 \end{aligned} \tag{15. 12}$$

Then, the square of the second moment will be

$$m_2^2(x) = m_2^2(p) + 2m_2(p)\, m_2(r) + m_2^2(r) \tag{15. 13}$$

Substituting Eqs. (15. 11) and (15. 13) into Eq. (15. 9), and noting the relationships presented in Eq. (15. 8), the following result is obtained

$$\text{Var}\,(\overline{x^2}) = \frac{2\sigma^2(r)\left[2\sigma^2(p) + \sigma^2(r)\right]}{2BT_1} \tag{15. 14}$$

In terms of the normalized variance ϵ^2, Eq. (15. 14) becomes

$$\epsilon^2 = \frac{\mathrm{Var}\,(\overline{x^2})}{\left[\sigma_x^2\right]^2} = \frac{\sigma^2(r)\left[2\sigma^2(p) + \sigma^2(r)\right]}{BT_1\left[\sigma^2(p) + \sigma^2(r)\right]^2} \qquad (15.\ 15)$$

For simplicity, let the term (p/r) denote the ratio of mean square values for the periodic and random portions. That is,

$$\frac{p}{r} = \frac{\sigma^2(p)}{\sigma^2(r)} \qquad (15.\ 16)$$

Now Eq. (15. 15)may be written as follows.

$$\epsilon^2 = \frac{\left(2\,\frac{p}{r} + 1\right)}{BT_1\left(\frac{p}{r} + 1\right)^2} \qquad (15.\ 17)$$

Note that if the signal x(t) is only periodic, the ratio p/r approaches infinity and $\epsilon^2 = 0$ as it should, since there is no variability associated with the measurement of mean square values for periodic signals. Also, if x(t) is only random, the ratio p/r is zero and $\epsilon^2 = 1/BT_1$, which is the correct normalized variance for mean square values associated with a random signal, as defined in Eq. (15. 5). For clarity, let the normalized variance for a measured mean square value of a random signal alone be denoted by ϵ_o^2. That is,

$$\epsilon_o^2 = \frac{1}{BT_1} \qquad (15.\ 18)$$

Substituting Eq. (15. 18) into Eq. (15. 17) gives the following relationship.

$$\epsilon^2 = \frac{\left(2\,\frac{p}{r} + 1\right)\epsilon_o^2}{\left(\frac{p}{r} + 1\right)^2} \qquad (15.\ 19)$$

Equation (15. 19) lays the groundwork for a statistical test for randomness. Assume a collection of N independent mean square values, $\overline{x_i^2}$ (i = 1, 2, 3, . . . , N), are measured from a stationary vibration

response signal. Each measured value $\overline{x_i^2}$ constitutes an estimate of σ_x^2. This collection may be obtained from a single sample record of length T by averaging over each of N equally long segments with an averaging time of $T_1 = T/N$. The variance of the measured mean square values can be estimated from the sample variance s^2 as follows.

$$s^2 = \frac{1}{N}\sum_{i=1}^{N}(\overline{x_i^2} - \overline{x^2})^2 = \frac{1}{N}\sum_{i=1}^{N}(\overline{x_i^2})^2 - (\overline{x^2})^2 \qquad (15.20)^*$$

where

$$\overline{x^2} = \frac{1}{N}\sum_{i=1}^{N}\overline{x_i^2}$$

Here, $\overline{x^2}$ is the mean square value averaged over the collection length $T = NT_1$. Then, the normalized variance for a mean square value measurement may be estimated by

$$\hat{\epsilon}^2 = \frac{s^2}{(\overline{x^2})^2} \qquad (15.21)$$

In Eq. (15.21), both s^2 and $\overline{x^2}$ are random variables. However, it is shown in Section 13.5.3 that the variability of the term $(\overline{x^2})^2$ is small compared to the variability of s^2 if the quantity BT_1 is large, say greater than 10. In this case, it may be assumed that $\overline{x^2} = \sigma_x^2$ for the problem at hand.

The sample variance s^2 from Eq. (15.20) will have a distribution associated with the chi-squared distribution as follows.

$$\frac{s^2}{\mathrm{Var}\,(\overline{x^2})} \sim \frac{\chi^2}{N} \qquad (15.22)$$

where "$\sim$" means "distributed as", and χ^2 is a chi-squared distribution with (N - 1) degrees of freedom.

*A biased expression for s^2 is employed here so that all statistical procedures to follow will be consistent with procedures outlined in Ref. [1].

From the relationships in Eq. (15. 21), it follows that

$$\frac{\hat{\epsilon}^2}{\epsilon^2} \sim \frac{\chi^2}{N} \tag{15.23}$$

From Eq. (15. 23), the following probability statement may be made.

$$\text{Prob}\left[\frac{\hat{\epsilon}^2}{\epsilon^2} \geq \frac{\chi^2_{(1-\alpha)}}{N}\right] = (1 - \alpha) \tag{15.24}$$

where α is the level of significance.

Let it be hypothesized that a sampled vibration response is random. If this is true, $\epsilon^2 = \epsilon_o^2$, and the hypothesis H_0 is

$$H_0 : \hat{\epsilon}^2 = \epsilon_o^2 \tag{15.25}$$

If a periodic component is present, $\hat{\epsilon}^2$ will be equivalent to some ϵ^2 less than ϵ_o^2 as defined in Eq. (15. 19), which is why a one-sided probability statement is used in Eq. (15. 24). Then, H_0 may be tested by computing $\hat{\epsilon}^2$ from a collection of mean square measurements, computing ϵ_o^2 from the BT_1 product for the measurements, and comparing the ratio of $\hat{\epsilon}^2/\epsilon_o^2$ to the χ^2 limit in Eq. (15. 24) at any desired level of significance α . The region of acceptance for H_0 is

$$\frac{\hat{\epsilon}^2}{\epsilon_o^2} \geq \frac{\chi^2_{(1-\alpha)}}{N} \tag{15.26}$$

If $\hat{\epsilon}^2/\epsilon_o^2$ is greater than the above noted limit, H_0 is accepted and vibration response is considered random. If $\hat{\epsilon}^2/\epsilon_o^2$ is less than the noted limit, H_0 is rejected and there is reason to suspect that a periodic component is present in the vibration response. **A plot of Eq. (15. 26) for** $\alpha = 0.01$ and 0.05 is presented in Figure 15. 4.

The level of significance α is the probability of a Type I error for the test. The risk of a Type I error may be reduced by testing with a smaller value of α. Now consider the probability of a Type II error denoted by β. β is a function of α, N, and the deviation of ϵ^2 from ϵ_o^2 which one

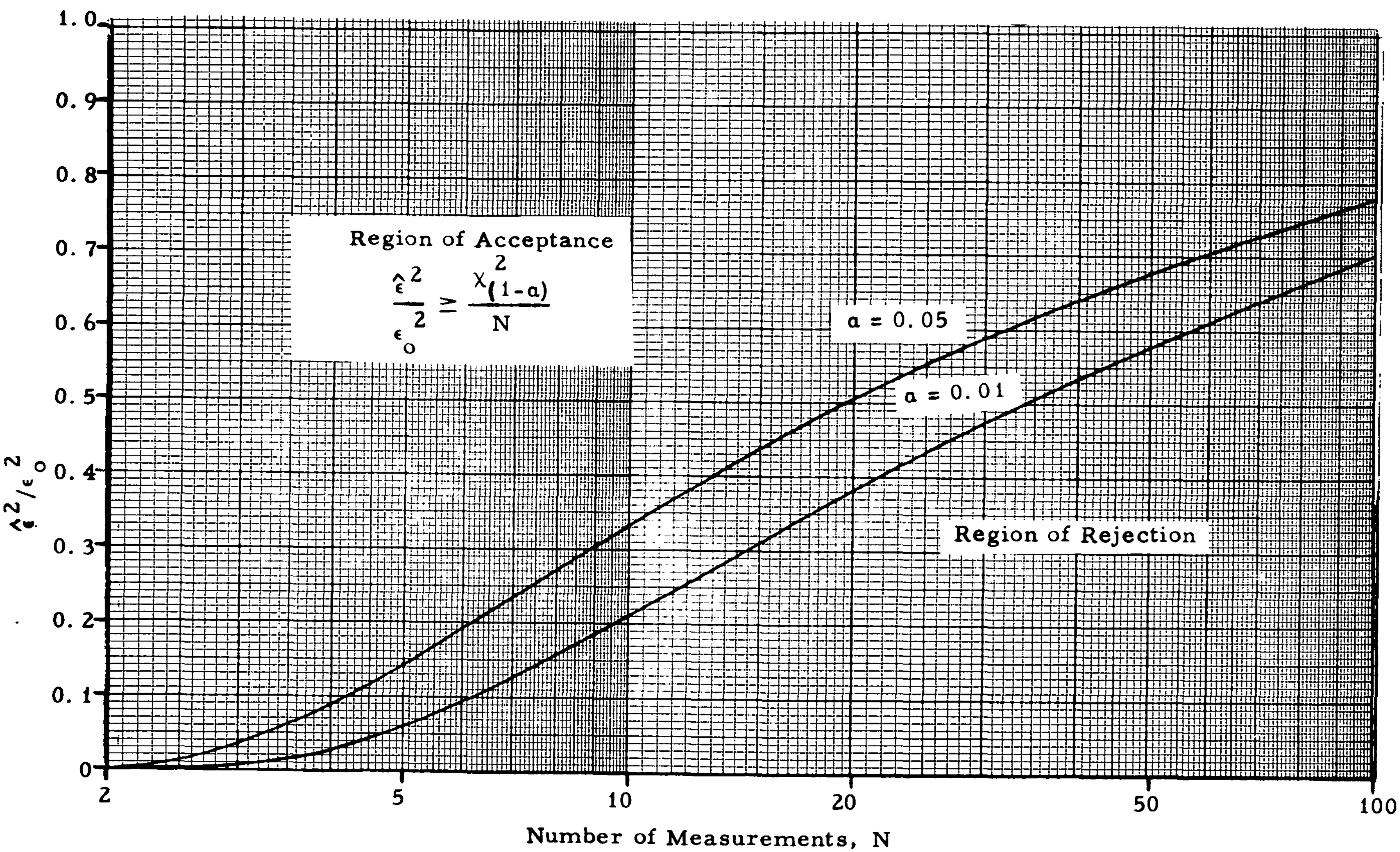


Figure 15.4 Acceptance Regions for Randomness Test B

wishes to detect. In general, both α and β can be reduced only by increasing the number of measurements N. The number N required for any value of α and β is given by the Operating Characteristic (OC) curves for Eq. (15. 26). The OC curves for $\alpha = 0.05$ and 0.01 are available in Figures 6. 19 and 6. 20 of Ref. [3]. Note that the referenced OC curves are developed for unbiased variance estimates. However, this does not affect the application of the curves to the problem here as long as the region of acceptance is consistent with the method of computing $\hat{\epsilon}^2$, as shown in Eq. (15. 26).

In summary, the procedure for applying Randomness Test B is as follows.

1. Determine the magnitude of a periodic component in terms of a p/r ratio which one wishes to detect with high probability.
2. Determine the ratio ϵ^2/ϵ_o^2 for that p/r ratio from Eq. (15. 19).
3. Determine the desired values for α and β, the risk of making a Type I and Type II error.
4. Determine the number of measurements N required from an OC curve for a one-sided (lower tail) χ^2 test.
5. Obtain the required number of mean square measurements, $\overline{x_i^2}$, and compute $\hat{\epsilon}^2$ from Eq. (15. 21).
6. Compute ϵ_o^2 from Eq. (15. 18).
7. Test the ratio $\hat{\epsilon}^2/\epsilon_o^2$ against $H_0 : \hat{\epsilon}^2 = \epsilon_o^2$.

It should be noted that the effectiveness of the procedure will increase as the bandwidth B is made small. Thus, if a sharp peak is observed in the power spectrum, the frequency range of the peak should be isolated by a narrow band filter. If the peak represents a sine wave, the random portion within the narrow bandwidth B will be very small, causing a large p/r ratio. This not only improves the power of the test, but it helps to define the value of B needed to compute ϵ_o^2 .

Example:

The above discussions will be clarified by considering an example. Assume that a three-second long sample record is obtained from a stationary vibration response signal. The vibration signal is to be tested for randomness at the 5% level of significance by investigating the variance

of mean square measurements. Assume that a power spectrum measured from the sample record using a 50 cps wide filter reveals a peak with a magnitude that is about ten times greater than the power spectral density at neighboring frequencies. The frequency range of the test for randomness should be limited to this suspicious peak.

Assume the 50 cps narrow band pass filter is tuned over this 10:1 peak. Let it be hypothesized that the signal within this bandwidth is random. It is desired that a p/r ratio of ten be detected with a Type II error probability of $\beta = 0.05$. From Eq. (15. 19)

$$\frac{\epsilon^2}{\epsilon_o^2} = \frac{(20 + 1)}{(10 + 1)^2} = 0.174 \text{ for } \frac{p}{r} = 10$$

From Figure 6. 19 of Ref. [3], for $(\epsilon^2/\epsilon_o^2) = 0.174$ ($\lambda = \epsilon/\epsilon_o = 0.42$) and $\alpha = \beta = 0.05$, the required number of mean square measurements is N = 9.

The three-second long sample record is now divided into nine equal segments, each $T_1 = 0.33$ seconds long. Note that $BT_1 = 16.5 > 10$. The mean square level for each segment is measured and the estimated normalized variance $\hat{\epsilon}^2$ is computed using Eq. (15. 21). From Eq. (15. 26), for N = 9, the region of acceptance is

$$\frac{\hat{\epsilon}^2}{\epsilon_o^2} \geq 0.332$$

From Eq. (15. 18), $\epsilon_o^2 = 0.0606$, so the acceptance region for $\hat{\epsilon}^2$ becomes

$$\hat{\epsilon}^2 \geq 0.0201$$

If $\hat{\epsilon}^2$ is greater than the above limit, the hypothesis of randomness is accepted. If $\hat{\epsilon}^2$ is less than the above limit the hypothesis of randomness is rejected and there is reason to suspect that the peak in the power spectrum represents a sine wave.

15.1.4 Bandwidth and Averaging Time Considerations

Randomness Test A, as originally proposed in Ref. [1], requires that the equivalent ideal lower and upper cutoff frequencies, f_a and f_b, be known for the signal under investigation. The method for computing f_a and f_b is developed and presented in Section 12.3.2.

Randomness Test B proposed here requires that the normalized variance ϵ_o^2 be known for the signal under investigation (when that signal is assumed to be random). From Eq. (15.18), ϵ_o^2 is a function of the equivalent ideal signal bandwidth B and the averaging Time T_1 for each mean square estimate. Bandwidth and averaging time considerations are discussed in Section 13.1.3. However, some additional discussion is warranted here.

(a) Bandwidth Considerations

The general procedures for computing an equivalent ideal bandwidth for a filtered random signal are discussed in Section 13.1.3, and in greater detail in Section 16.1.5(a). For the problem at hand, Randomness Test B is most effective when applied to isolated peaks in the power spectrum of the signal being studied. Each peak may represent a sine wave or a narrow band random signal.

It is assumed for application of the test that a peak represents a narrow band random signal. A narrow band filter, such as the filter incorporated in a power spectral density analyzer, may be tuned over a power spectral density peak to isolate it from the remainder of the signal. The filter bandwidth B should be selected so that it is less than the apparent width of the peak as indicated from a power spectral density plot. Naturally, if the peak were a sine wave, the apparent width would be equal to the bandwidth of the filter used for the power spectrum measurement. In any case, the bandwidth B needed for Eq. (15.18) may now be assumed to be the bandwidth of the narrow filter employed. This greatly simplifies the application of Randomness Test B to real data.

(b) Averaging Considerations

The ideal method of averaging is by linear integration over the desired time interval, T_1. However, averaging is often accomplished by continuous smoothing with a low pass RC filter having a time constant K.

The general associations between the time constant K and the equivalent averaging time T_1 are discussed in Section 13. 1. 3. There are special considerations which apply to the present problem.

Assume a long record of length T is to be tested for randomness by analysis of a continuous RC averaged mean square estimate as shown in Figure 15. 5.

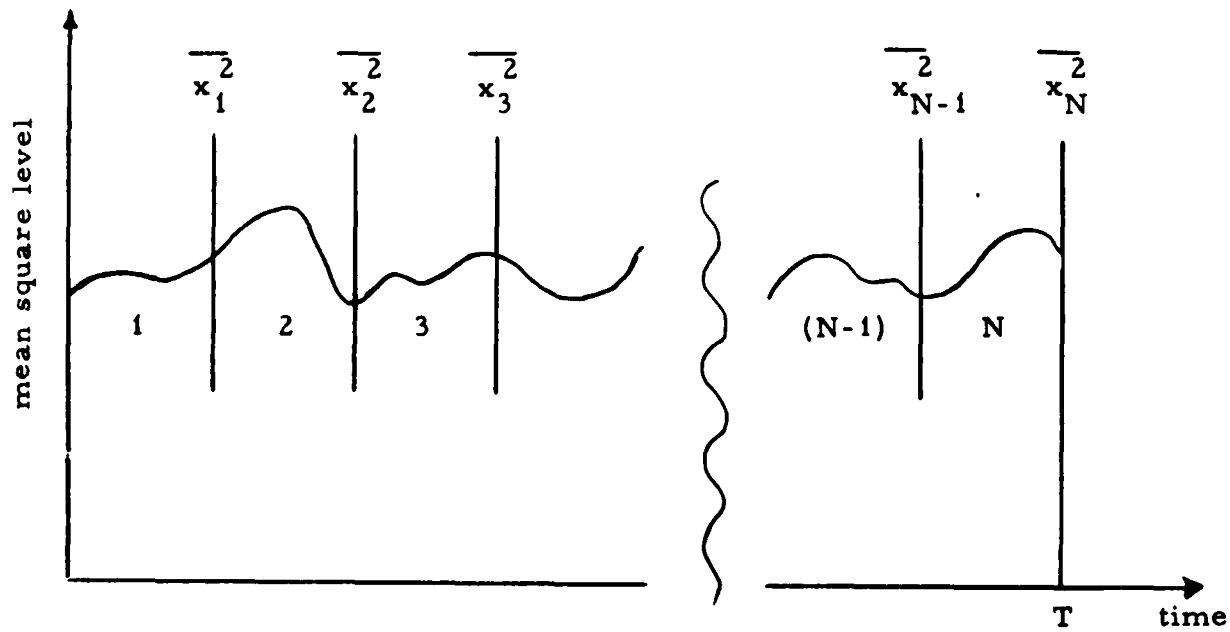


Figure 15. 5 Analysis of Continuous Mean Square Levels

Let the time constant K be short as compared to T; that is, $K \ll T$. Now divide the continuous mean square estimate into N equal intervals such that each interval is about 3K or 4K long. The level of the continuous mean square estimate at the end of each interval will constitute a mean square measurement $\overline{x_i^2}$ based on an equivalent averaging time of $T_1 = 2K$. The individual readings should be 3K or 4K apart to assure that they are statistically independent. Note that the points at which readings are made can also be selected by random sampling. Under no circumstances, however, should the points be selected without a pre-designated plan (equal intervals or random intervals). Otherwise, the person reducing the data may bias the results.

15.2 DESIGN OF EXPERIMENTS AND PROCEDURES

15.2.1 General Design and Procedures

The general purpose of these experiments is twofold. The first is to study the practicability of Randomness Test A. The foundation of Test A is based upon zero crossing theory which is substantiated in Section 12. The second objective is to verify the theoretical foundation for Randomness Test B.

No additional data over that presented in Section 12 is needed to accomplish the first objective. Thus, all the new experiments performed in this section are concerned with Randomness Test B.

15.2.2 Procedure for Verifying Randomness Test B

The objective is to confirm the validity of Eq. (15.19), which is the basis for Randomness Test B. The verification procedure is to collect N number of mean square measurements from a signal consisting of a sine wave in a random signal background. An estimated normalized variance $\hat{\epsilon}^2$ is computed using Eqs. (15.20) and (15.21). The resulting value for $\hat{\epsilon}^2$ is then tested for equivalence to the theoretical value for ϵ^2 computed using Eq. (15.19). The experiment is repeated for several different (p/r) ratios. Note that the bandwidth and averaging time relationships associated with the value of ϵ_o^2 in Eq. (15.19) are fully substantiated by experiments performed in Section 13. Thus, it is considered sufficient here to study Eq. (15.19) for only one set of values for B and T_1.

More specifically, a random signal with a bandwidth of B = 56 cps on a center frequency of f_c = 1000 cps is mixed with a sine wave having a frequency of f_0 = 1000 cps. A total of N = 61 consecutive mean square measurements are obtained by averaging with a low pass RC filter having a time constant of K = 0.16 seconds (T_1 = 0.32 seconds). Thus, from Eq. (15.18), $\epsilon_o^2 = (1/BT_1) = 0.056$.

The experiment is performed for five different p/r ratios producing five different expected values for ϵ^2 from Eq. (15.19) as follows.

1. p/r = 0 ; $\epsilon^2 = 0.056$
2. p/r = 1 ; $\epsilon^2 = 0.042$
3. p/r = 2 ; $\epsilon^2 = 0.031$
4. p/r = 4 ; $\epsilon^2 = 0.020$
5. p/r = 9 ; $\epsilon^2 = 0.011$

For each experiment, $\hat{\epsilon}^2$ is computed from the N = 61 mean square measurements using Eq. (15. 20) and (15. 21). The values for $\hat{\epsilon}^2$ are effectively variance estimates (since $BT_1 \gg 15$), and will have a sampling distribution associated with the χ^2 distribution as given in **Eq. (15. 23).** Thus, the following probability statement may be made.

$$\text{Prob}\left[\frac{\chi^2_{(1-\alpha/2)}}{N} \leq \frac{\hat{\epsilon}^2}{\epsilon^2} \leq \frac{\chi^2_{\alpha/2}}{N}\right] = (1 - \alpha) \qquad (15.\ 27)$$

For each of the five (p/r) ratios, let it be hypothesized that the experimentally determined value $\hat{\epsilon}^2$ is equivalent to the theoretical value ϵ^2 predicted by Eq. (15. 19). This hypothesis, H_0, is to be tested at the $\alpha = 0.05$ level of significance. Then, from **Eq. (15. 27)**, the region of acceptance will be as follows.

$$0.66 \leq \frac{\hat{\epsilon}^2}{\epsilon^2} \leq 1.37 \qquad (15.\ 28)$$

If the experimental to theoretical variance ratio falls within the above noted limits, H_0 is accepted and Eq. (15. 19) is considered to be valid. If the ratio falls outside the noted limits, H_0 is rejected and there is reason to question the practical validity of Eq. (15. 19).

Note that the test for equivalence used here is exactly the same as is employed for the experiments in Section 13. The Type I and Type II error considerations are as developed and discussed in Section 13. 2. 3. The probability of a Type I error for the test of H_0 is at least $\alpha = 0.05$. The probability of a Type II error for the test of H_0 is at least $\beta = 0.05$ for detecting a 2:1 difference between $\hat{\epsilon}^2$ and ϵ^2 (a 1.4:1 difference between $\hat{\epsilon}$ and ϵ).

15.3 INSTRUMENTATION

15.3.1 Instruments and Test Set-up

The laboratory instruments employed for these experiments are listed in Table 15.1. A block diagram for the test set-up is illustrated in Figure 15.6. Except for the random noise generator, all instruments are the property of the Norair Division of Northrop Corporation, and were in current calibration at the time of the experiments.

The random noise generator (Item A) is used as the source of a signal which is considered in these experiments to be a stationary random signal. The detailed characteristics of this instrument are presented in Section 14.3.

The sine wave generator (Item B) is used as a source of a signal which is known to be nonrandom. This item is also employed for calibration purposes as discussed in Section 15.3.2. The frequency counter (Item C) determines the frequency of the sinusoidal signal with an accuracy of ± 1 cps.

The sine wave - random noise mixer (Item D) is used to create a controlled nonrandom signal with a defined (p/r) ratio. The narrow band pass filter (Item E) is employed to band limit the frequency range of the signal to be investigated. The characteristics and calibration of this filter are discussed in Section 15.3.2. The voltmeter (Item F) measures the rms voltage level of the signal to be investigated.

The mean square level analyzer (Item G) is used to obtain a continuous RC averaged mean square level estimate. This instrument is a true RMS voltmeter with an analog mean square level output. The strip chart recorder (Item H) records the mean square level time history samples for the signal to be investigated.

15.3.2 Calibration of Test Set-up

The exactness of the experiments is dependent primarily upon the accurate knowledge of two parameters, namely the bandwidth B and the averaging time T_1. The filter employed to obtain the narrow bandwidth of B = 56 cps is the same filter used for some of the experiments in Section 13. The characteristics of this filter are discussed in Section 13.3.2. The time constant of K = 0.16 seconds for the mean square level circuit of Item G is determined using the calibration procedures detailed in Section 13.3.3.

Item	Description	Manufacturer	Model No.	Serial No. *
A	Random Noise Generator	General Radio Co.	1390A	937
B	Audio Oscillator	Hewlett-Packard	202D	67117
C	Universal EPUT and Timer	Berkeley Division	7350	79528
D	Sine Wave-Random Noise Mixer	Northrop Corporation	----	---
E	Narrow Band Pass Filter	Technical Products Co.	**	NC 44065
F	True RMS Voltmeter	B & K Instruments	2409R	PR-13147-1
G	Mean Square Level Analyzer (True RMS Voltmeter)	Ballantine Laboratories, Inc.	320	1149
H	Strip Chart Recorder	Brush Instruments	Mark II	270822

*Except for Item A, the serial numbers refer to Northrop Corporation identification tags.

**This filter is part of a TPC power spectral density analyzer. See Table 13.1.

Table 15.1 Instruments Employed for the Experiments

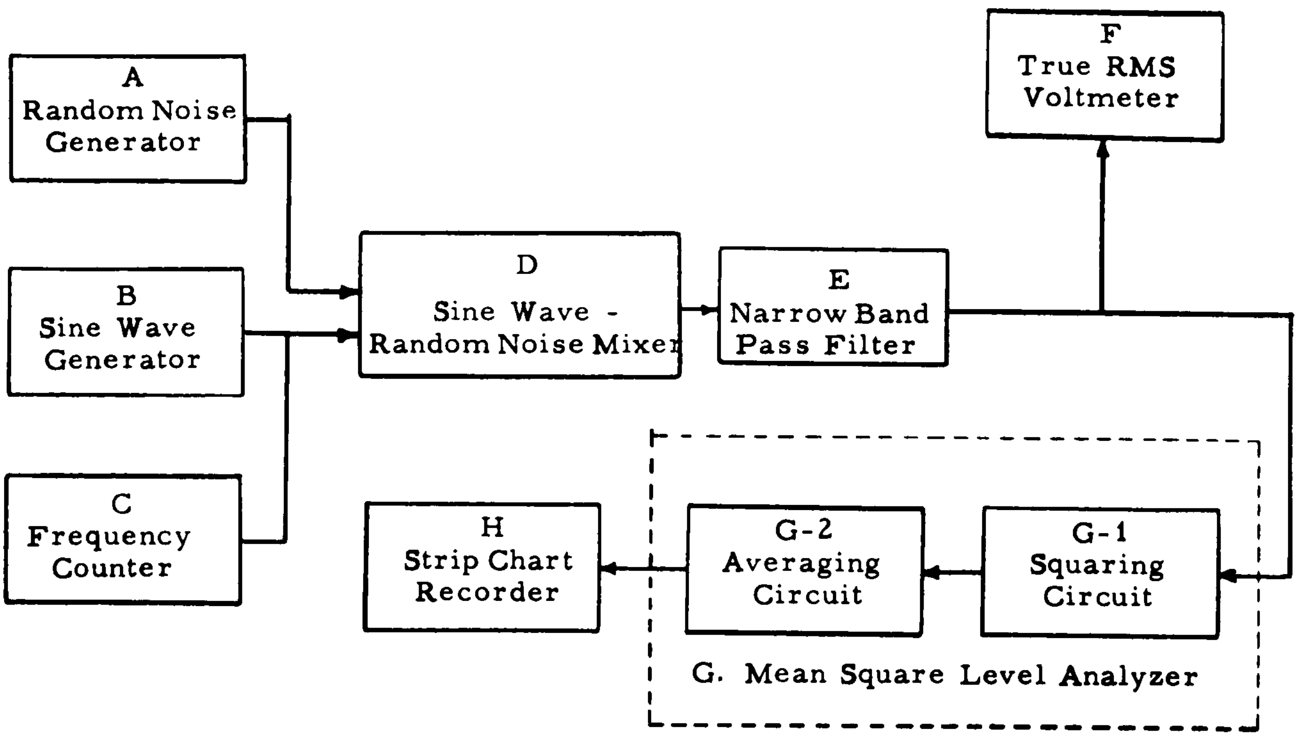


Figure 15.6 Block Diagram of Test Set-Up

The required (p/r) ratio for the signal to be investigated is established as follows. The random signal alone is applied from Item A and adjusted to read out the desired level at Item F, which has a long time constant that minimizes uncertainty fluctuations. The gain setting on Item A required to obtain the desired level is noted. The sine wave alone is then applied from Item B and adjusted to read out the desired level at Item F. The gain setting of Item B required to obtain the desired level is noted. The sine wave-random signal combination is obtained by adjusting the gain settings on Items A and B to the previously noted positions. The individual signals as well as the combined signal are also recorded by Item H as a check.

15.4 RESULTS OF EXPERIMENTS

15.4.1 Mean Square Value Data

The 61 mean square values gathered for each of the five (p/r) ratios are presented in Table 15.2.

15.4.2 Sampling Statistics for Mean Square Value Estimates

The sample mean and variance for the 61 mean square values gathered for each of the five experiments are presented in Table 15.3. The sample mean and variance are computed using Eq. (15.20).

Mean Square Value Estimates $\overline{x_i^2}$ for Ratios of $(p/r) = \sigma^2(p)/\sigma^2(r)$

f_c = 1000 cps; B = 56 cps; K = 0.16 (T_1 = 0.32); f_0 = 1000 cps

Estimate Number	1 (p/r)=0	2 (p/r)=1	3 (p/r)=2	4 (p/r)=4	5 (p/r)=9	Estimate Number	1 (p/r)=0	2 (p/r)=1	3 (p/r)=2	4 (p/r)=4	5 (p/r)=9
1	2.07	1.12	2.36	2.07	1.95	31	1.84	1.70	2.01	2.79	1.91
2	2.52	2.37	2.10	1.82	2.13	32	1.56	1.53	1.86	2.08	1.68
3	2.22	1.18	2.31	2.40	2.09	33	2.19	2.03	1.93	1.73	1.82
4	1.76	1.60	1.83	2.18	1.62	34	2.60	1.90	3.20	2.48	2.03
5	1.80	1.25	1.72	2.16	1.72	35	1.37	1.41	1.98	1.72	2.07
6	2.25	2.09	2.14	1.89	1.88	36	2.84	2.14	1.69	2.11	1.90
7	1.79	1.79	1.49	1.78	1.81	37	1.40	2.22	2.04	1.82	1.53
8	2.81	2.32	1.59	2.54	1.86	38	1.83	1.43	1.76	1.73	1.55
9	2.08	1.98	2.12	2.00	1.87	39	1.09	2.13	2.01	2.10	2.25
10	2.80	1.61	1.30	2.04	2.06	40	2.27	1.57	2.28	1.90	2.08
11	1.40	1.26	1.59	1.96	1.91	41	2.33	2.38	2.45	2.20	1.93
12	2.09	2.07	2.08	2.23	1.88	42	2.70	1.64	1.78	2,27	1.92
13	2.45	1.92	1.79	2.18	1.73	43	2.01	1.98	1.49	2.04	1.91
14	1.97	2.48	1.91	2.40	1.86	44	1.62	2.13	1.68	1.88	2.26
15	2.20	2.29	2.01	1.62	2.27	45	3.08	1.62	1.41	1.91	1.76
16	2.19	2.09	2.49	2.10	2.03	46	1.76	2.42	1.78	1.66	1.82
17	1.78	2.00	2.11	2.22	1.95	47	1.49	1.39	2.17	2.20	2.01
18	2.13	1.84	2.12	2.41	2.24	48	2.39	1.22	2.29	1.90	1.80
19	1.63	2.03	1.61	2.21	1.49	49	2.10	1.75	1.66	1.99	1.89
20	2.21	2.50	1.85	2.13	1.82	50	1.94	1.69	2.19	2.08	2.05
21	1.78	1.73	1.94	1.73	2.22	51	2.21	1.81	1.78	1.89	1.89
22	2.12	1.72	1.74	1.82	1.76	52	1.80	1.10	2.00	1.96	2.03
23	1.50	1.85	2.02	1.88	1.82	53	2.21	1.70	1.93	2.49	2.22
24	2.00	1.52	1.69	1.81	1.73	54	1.79	2.06	2.24	2.11	2.19
25	3.33	1.38	2.13	2.50	1.62	55	1.61	2.87	2.75	1.89	1.78
26	2.33	1.92	2.98	2.11	1.85	56	1.64	2.82	1.72	2.17	2.13
27	1.82	1.97	1.79	1.79	2.00	57	2.08	1.30	1.76	2.52	2.01
28	2.22	1.85	2.88	1.63	1.91	58	1.78	1.79	2.22	1.94	2.05
29	2.49	1.69	1.70	2.04	1.88	59	2.49	1.97	1.63	2.09	1.71
30	2.42	1.80	2.31	1.82	2.07	60	1.45	2.63	1.98	2.24	1.83
						61	1.83	2.01	2.12	1.68	1.72

Table 15.2 Mean Square Value Estimates for Various (p/r) Ratios

Sample Mean $\overline{x^2}$ and Sample Variance s^2 For Mean Square Value Estimates $\overline{x_i^2}$			
Description of Experiment	Sample Size N	Sample Mean $\overline{x^2}$	Sample Variance s^2
1. (p/r) = 0	61	2.057	0.192
2. (p/r) = 1	61	1.862	0.162
3. (p/r) = 2	61	2.000	0.129
4. (p/r) = 4	61	2.050	0.0653
5. (p/r) = 9	61	1.914	0.0341

Table 15.3 Sample Means and Variances for Mean Square Value Estimates

15.4.3 Results of Statistical Tests

Tests of the null hypothesis, $H_0 : \hat{\epsilon}^2 = \epsilon^2$, are presented in Table 15.4. The hypothesis H_0 is accepted for all five experiments with a minimum probability of a Type I and Type II error of 5%. Thus, there is no reason to question the validity of Eq. (15.19) for the range of (p/r) ratios tested.

Statistical Test of Hypothesis $H_0 : \hat{\epsilon}^2 = \epsilon^2$				
Experiment	ϵ^2 from Eq. (15.19)	$\hat{\epsilon}^2 = \dfrac{s^2}{(\overline{x^2})^2}$	$\dfrac{\hat{\epsilon}^2}{\epsilon^2}$	Region of Acceptance for H_0 $0.66 \leq \dfrac{\hat{\epsilon}^2}{\epsilon^2} \leq 1.37$
1. (p/r)=0	0.056	0.045	0.80	accepted
2. (p/r)=1	0.042	0.047	1.1	accepted
3. (p/r)=2	0.031	0.032	1.0	accepted
4. (p/r)=4	0.020	0.016	0.80	accepted
5. (p/r)=9	0.011	0.0093	0.84	accepted

Table 15.4 Statistical Tests for Equivalence

15.5 DISCUSSION OF RESULTS

15.5.1 Randomness Test A

The effectiveness of Randomness Test A may be studied using the zero crossing data presented in Section 12. Consider first the zero crossings measured for random signals with uniform power spectra. This data is designated as Area A in Section 12 and is presented in Table 12. 2. Apply Randomness Test A to this data (A-1 through A-3 in Table 12. 2) at the $\alpha = 0.05$ level of significance.

A-1. Sample records with length T = 0.03 seconds and frequency range $f_a = 28.5$ cps to $f_b = 2100$ cps, giving n = 146.49 events per record.

From Eq. (15. 1), the sampling distribution of zero crossings will have a mean and variance of

$$\mu_{\nu_0} = 73.24$$

$$\sigma^2_{\nu_0} = 36.37$$

For a 5% level of significance, the acceptance region for the number of zero crossings in a sample record from a random signal is

$$\mu_{\nu_0} \pm 1.96\sigma_{\nu_0} = 73.24 \pm 11.82$$

Then, if it is hypothesized that each sample record is obtained from a random signal, the hypothesis will be accepted at the 5% level of significance if the number of zero crossings is between

$$62 \leq \nu_0 \leq 85$$

From Table 12. 2, for the 31 sample records gathered, the hypothesis of randomness is accepted for 29 of the sample records, and erroneously rejected for two of the sample records (Samples No. 3 and 28). Two Type I errors out of 31 tests at the 5% level of significance is what one would anticipate.

A-2. Sample records with length T = 0.05 seconds and frequency range f_a = 28.5 cps to f_b = 1050 cps, giving n = 122.92 events per record.

For this case, the sampling distribution of zero crossings will have a mean and variance of

$$\mu_{\nu_0} = 61.46$$

$$\sigma^2_{\nu_0} = 30.48$$

For a 5% level of significance, the acceptance region for the number of zero crossings in a sample record from a random signal is

$$\mu_{\nu_0} \pm 1.96\sigma_{\nu_0} = 61.46 \pm 10.82$$

Then a hypothesis of randomness will be accepted at the 5% level of significance if the number of zero crossings is between

$$51 \leq \nu_0 \leq 72$$

From Table 12.2, for the 31 sample records gathered, the hypothesis of randomness is accepted for 30 of the sample records, and erroneously rejected for one sample record (Sample No. 6). Once again, one Type I error out of 31 tests at the 5% level of significance is consistent.

A-3. Sample records with length T = 0.10 seconds and frequency range f_a = 95 cps to f_b = 630 cps, giving 157.61 events per record.

For this case, the sampling distribution of zero crossings will have a mean and variance of

$$\mu_{\nu_0} = 78.80$$

$$\sigma^2_{\nu_0} = 39.14$$

For the 5% level of significance, the acceptance region for the number of zero crossings in a sample record from a random signal is

$$\mu_{\nu_0} \pm 1.96\,\sigma_{\nu_0} = 78.80 \pm 12.27$$

Then a hypothesis of randomness will be accepted at the 5% level of significance if the number of zero crossings is between

$$67 \leq \nu_0 \leq 91$$

From Table 12.2, for the 31 sample records gathered, the hypothesis of randomness would be accepted for 28 of the sample records, and erroneously rejected for three sample records (Samples No. 11, 9, and 20). Three Type I errors out of 31 tests at the 5% level of significance is reasonable.

Consider next the zero crossings measured for nonrandom signals consisting of sine wave-random combination signals. This data is designated as Area C in Section 12 and is presented in Table 12.3. Apply Randomness Test A to this data (C-1 through C-9 in Table 12.3) at the $\alpha = 0.05$ level of significance. Note that for this illustration, the Type II error of the test is of primary importance, since it is desired to demonstrate the ability of the test to reject a hypothesis of randomness for signals which are truly nonrandom. The number of events represented by each sample record in Table 12.3 is less than $n = 160$. For a test at the 5% level of significance, the Type II error would be unreasonably large. However, to facilitate the desired illustrations, a sample record representing a much larger number of events can be created by simply adding together all 31 sample records gathered under each Test Procedure to form one long record for each experiment.

C-1 through C-4. Sample records with length $T = 1.55$ seconds and frequency range $f_a = 28.5$ cps to $f_b = 1050$ cps, giving $n = 3810.6$ events per record. Sine wave present with $(p/r) = 1.0$.

From Eq. (15.1) the sampling mean and variance for a random signal would be as follows:

$$\mu_{\nu_0} = 1905.3 \quad ; \quad \sigma^2_{\nu_0} = 952.4$$

The acceptance region for testing at the 5% level of significance is

$$\mu_{\nu_0} \pm 1.96\sigma_{\nu_0} = 1905 \pm 60$$

Then the hypothesis of randomness will be accepted if the number of zero crossings in a sample record is

$$1845 \leq \nu_0 \leq 1965$$

The probability of a Type II error for the test is $\beta = 0.05$ for detecting a difference of 6.3% in the expected number of zero crossings.

From Table 12.3, the total number of zero crossings for each experiment (T = 1.55 seconds) are as follows:

Test Procedure C-1 (f_0 = 200 cps) ; ν_0 = 1817 crossings

Test Procedure C-2 (f_0 = 400 cps) ; ν_0 = 1993 crossings

Test Procedure C-3 (f_0 = 600 cps) ; ν_0 = 2080 crossings

Test Procedure C-4 (f_0 = 800 cps) ; ν_0 = 2188 crossings

Then the randomness hypothesis would be correctly rejected for all four sample records tested.

C-5 through C-9. Sample records with length T = 3.10 seconds and frequency range f_a = 95 cps to f_b = 630 cps, giving n = 4885.6 events per record. Sine wave present with (p/r) = 0.25.

From Eq. (15.1) the sampling mean and variance for a random signal would be as follows:

$$\mu_{\nu_0} = 2442.8 \quad ; \quad \sigma^2_{\nu_0} = 1221.1$$

The acceptance region for testing at the 5% level of significance is

$$\mu_{\nu_0} \pm 1.96\sigma_{\nu_0} = 2443 \pm 68$$

Then the hypothesis of randomness will be accepted if the number of zero crossings in a sample record is

$$2375 \leq \nu_0 \leq 2511$$

The probability of a Type II error for the test is $\beta = 0.05$ for detecting a difference of 5.6% in the expected number of zero crossings.

From Table 2, the total number of zero crossings for each experiment ($T = 3.10$ seconds) are as follows:

Test Procedure C-5	($f_0 = 150$ cps) ;	$\nu_0 = 1849$ crossings
Test Procedure C-6	($f_0 = 250$ cps) ;	$\nu_0 = 2088$ crossings
Test Procedure C-7	($f_0 = 350$ cps) ;	$\nu_0 = 2410$ crossings
Test Procedure C-8	($f_0 = 450$ cps) ;	$\nu_0 = 2730$ crossings
Test Procedure C-9	($f_0 = 550$ cps) ;	$\nu_0 = 3184$ crossings

Then, the randomness hypothesis is correctly rejected for all but one of the five sample records tested, and erroneously accepted for one sample record (Test Procedure C-7).

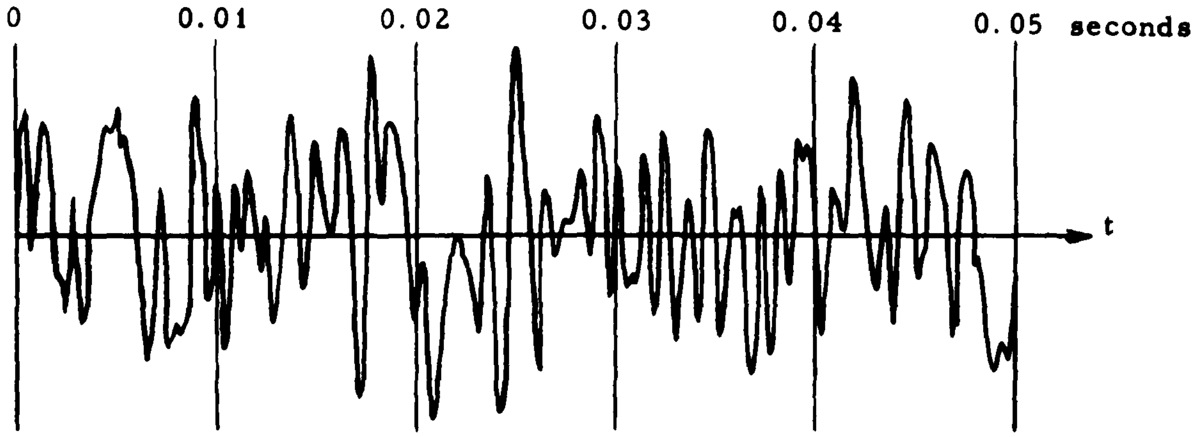
The power with which Random Test A detects nonrandomness is clearly shown by the above illustrations. A segment of an amplitude time history record used to obtain data for the above illustrations is shown in Figure 15.7. A random signal record is shown with the nonrandom record. Note that the presence of a periodic component is not readily apparent from the signal time history. This should give greater appreciation for the power with which the test detects nonrandomness.

Consider now the zero crossings measured for signals which are random but do not have uniform power spectra. This data is designated as Area B in Section 12 and is presented in Table 12.2. Apply Randomness Test A to this data (B-1 through B-4 in Table 12.2) at the $\alpha = 0.05$ level of significance.

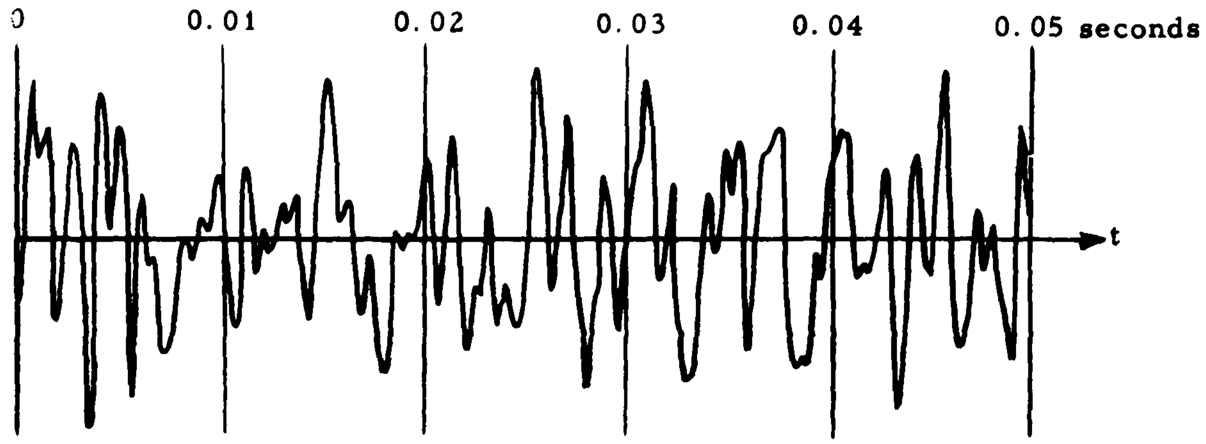
B-1 through B-5. Sample records with length $T = 0.10$ seconds and frequency range $f_a = 95$ cps to $f_b = 630$ cps, giving $n = 157.61$ events per record. Narrow peak present in the power spectrum with magnitude of 100:1.

For this case the region of acceptance for testing a hypothesis of randomness is exactly the same as for the data under A-3. Then, a hypothesis of randomness will be accepted at the 5% level of significance if the number of zero crossings in a sample record is

$$67 \leq \nu_0 \leq 91$$



Random Signal: f_a = 28.5 cps, f_b = 1050 cps



Random Signal+ Sine Wave; f_a = 28.5 cps,
f_b = 1050 cps, f_o = 200 cps. Mean square
value of sine wave is 1/4 mean square
value of random component; that is, (p/r) = 0.25

Figure 15.7 Typical Amplitude Time History Records

From Table 12. 2, for a narrow peak in the power spectrum at 150 cps (Test Procedure B-1), the hypothesis of randomness would be accepted for only five sample records (Samples No. 1, 3, 8, 24, and 29), and would be erroneously rejected for the remaining 26 sample records.

For a narrow peak in the power spectrum at 350 cps (Test Procedure B-2), the hypothesis of randomness would be accepted for all 31 sample records. For this case, the actual sampling mean is not very different from that predicted by Eq. (15. 1), while the actual sampling variance is much less than predicted by Eq. (15. 1).

For a narrow peak in the power spectrum at 550 cps (Test Procedure B-3), the hypothesis of randomness would be erroneously rejected for all 31 sample records. Here the actual sampling mean is quite different from that predicted by Eq. (15. 1).

For three narrow peaks in the power spectrum at 150 cps, 350 cps, and 550 cps (Test Procedure B-4), the hypothesis of randomness would be accepted for 17 sample records (Samples No. 1, 2, 4, 5, 6, 9, 10, 11, 12, 14, 15, 19, 20, 24, 25, 27, and 28), and erroneously rejected for the remaining 14 sample records.

It is clear from the above analysis that Randomness Test A does not correctly detect randomness in signals which do not have reasonably uniform power spectra.

15. 5. 2 Randomness Test B

The effectiveness of Randomness Test B may be studied using the mean square value data presented in Section 15. 4. Apply Randomness Test B to each collection of 61 mean square values gathered for the five different signals (Experiments 1 through 5 in Table 15. 2) at the $\alpha = 0.05$ level of significance.

If it is hypothesized that the mean square values were measured from a random signal, from Eq. (15. 26) for $N = 61$ and $\alpha = 0.05$, the region of acceptance for the hypothesis is

$$\frac{\hat{\epsilon}^2}{\epsilon^2} \geq 0.71$$

From Eq. (15. 18), $\epsilon_o^2 = 0.056$. Thus, the hypothesis will be accepted at the 5% level of significance if the normalized variance for the measured mean square values is

$$\hat{\epsilon}^2 \geq 0.040$$

From Figure 6. 19 of Ref. [3], the probability of a Type II error for the test is $\beta = 0.05$ for detecting a true value of $\epsilon^2 = 0.55 \ \epsilon_o^2 = 0.0308$.

The normalized variance for each of the five collections of mean square values is presented in Table 15. 4. The results of the randomness test are as follows.

1. $(p/r) = 0$; $\hat{\epsilon}^2 = 0.045$; accepted
2. $(p/r) = 1$; $\hat{\epsilon}^2 = 0.047$; accepted
3. $(p/r) = 2$; $\hat{\epsilon}^2 = 0.032$; <u>rejected</u>
4. $(p/r) = 4$; $\hat{\epsilon}^2 = 0.016$; <u>rejected</u>
5. $(p/r) = 9$; $\hat{\epsilon}^2 = 0.0093$; <u>rejected</u>

The first experiment involves a signal which is truly random, so the acceptance is proper. The second experiment involves a nonrandom signal with a (p/r) ratio of unity. Thus, the acceptance here constitutes a Type II error. Note that the expected value of the normalized variance for $(p/r) = 1$ is $\epsilon^2 = 0.042 = 0.75\epsilon_o^2$. The probability of making a Type II error for this case is $\beta \approx 0.6$ or 60%. Hence, the occurrence of a Type II error for this case is not surprising. The last three experiments involve nonrandom signals with (p/r) ratios from 2 to 9. In all three cases, the hypothesis of randomness is correctly rejected.

15. 5. 3 Relative Effectiveness of Randomness Tests

It is clear from the discussions in Section 15. 5. 1 that Randomness Test A is exceptionally powerful under ideal conditions. The test detected a sine wave in a random signal with a (p/r) ratio of only 0. 25 using a sample record length of only 3. 10 seconds. Randomness Test B failed to detect a sine wave in a random signal with a (p/r) ratio of unity from 61 mean square values representing a total sample record length of 19. 5 seconds.

On the other hand, Randomness Test A may erroneously reject a random signal as being nonrandom if a sharp peak is present in the power

spectrum. In other words, Test A is unable to distinguish between sine waves and narrow band random signals. This is understandable since the test is based upon zero crossing information. A sine wave in an otherwise random signal and a sharp peak in the power spectrum of a random signal both tend to affect the resulting zero crossings in a similar manner. Thus, the only way Randomness Test A could be effectively used is to observe the power spectrum for the signal to be investigated, and apply the test to a frequency range where no sharp peaks are observed. However, as discussed in Section 15. 1. 1, if no distinct peaks are present in a properly resolved power spectrum, no periodic components could be present and further tests for randomness are unnecessary. Because narrow band random vibration responses are common for lightly damped flight vehicle structures, Randomness Test A does not appear to be practical for applications to flight vehicle vibration data.

Randomness Test B does not pose the problem discussed above. The theory of Test B is not dependent upon the shape of the power spectrum for the signal as long as an equivalent ideal bandwidth for the signal can be defined. In fact, Test B is most powerful when applied only to the narrow frequency range of sharp peaks observed in the power spectrum.

In summary, although Randomness Test A is more powerful than Randomness Test B under ideal conditions, Test A is considered impractical for applications to flight vehicle vibration data analysis.

15.6 CONCLUSIONS

Two different procedures for testing single sample records for randomness have been experimentally evaluated. One of the procedures (Test A) is the randomness test originally proposed in Section 6.1.5 of Ref. [1]. The second procedure (Test B) is based upon more recent concepts developed herein.

The experimental results confirm the validity of the theory for Randomness Test A, as developed in Ref. [1]. However, the results also indicate Test A will erroneously reject random signals as being non-random unless the power spectra of the signals are reasonably uniform. This fact makes the application of Test A to flight vehicle vibration response data impractical, since such data rarely have uniform power spectra. Randomness Test B proposed in this section does not present the above problem. Although Test B is less powerful, its application to flight vehicle vibration data is more practical. Thus, it is recommended that Randomness Test A from Ref. [1] be dropped in favor of the new test for randomness presented herein.

Qualitative techniques for evaluating the randomness of flight vehicle vibration data are also discussed. These techniques involve observation of certain distinguishing details in power spectral density functions, autocorrelation functions, and amplitude probability density functions. These qualitative procedures will often reveal periodicities in an otherwise random vibration without the need for quantitative testing of the data.

15.7 REFERENCES

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16. TESTS FOR STATIONARITY

16.1 THEORY OF TESTS FOR STATIONARITY

16.1.1 General Remarks

The general requirements for stationarity of a collection of random signal records forming a random process are discussed in Section 6.1.6 of Ref. [1]. As noted there, the term "stationary" usually applies to the properties of a collection (ensemble) of random signals determined by taking ensemble averages at specific times. However, the concept of stationarity may also be applied to the properties of a single random signal (assumed to exist over all time) determined by taking time averages over different time intervals. This concept of stationarity is called "self-stationarity" to distinguish it from the general concept of stationarity for an ensemble of random signals forming a random process.

For a single random signal, if the mean values, mean square values, and autocorrelation functions determined by taking time averages over different time intervals are statistically equivalent, the random signal is considered "weakly self-stationary." For a random signal representing a vibration response, the mean value (DC component) is usually zero. Furthermore, if the mean square values are equivalent, there is strong reason to believe that the autocorrelation functions are similar since autocorrelation functions are less than or equal to their associated mean square values. Hence, the determination of weak self-stationarity for random vibration signals can usually be limited to an examination of mean square levels. These matters are discussed in more detail in Section 6.1.6 of Ref. [1].

For simplicity in the present study, the term "stationary" will be used throughout the discussions where it is understood that only mean square values determined by time averaging are considered (weak self-stationarity).

A quantitative procedure for testing a single random signal record for stationarity is suggested in Section 6.1.8 of Ref. [1]. Experimental verification of that test procedure, hereafter referred to as Stationarity Test A, is a major subject of this section. Two other alternative tests for stationarity are also developed here and studied experimentally.

One of these alternative tests involves a simple chi-squared test of variances, and will hereafter be referred to as Stationarity Test B. The second alternate test makes use of the F_{max} statistic to test the maximum and minimum of a group of mean square estimates, and will hereafter be referred to as Stationarity Test C. These three tests are discussed in Sections 16. 1. 2 through 16. 1. 4.

All three procedures test for the same basic characteristic; namely, the equivalence of a collection of mean square estimates. The collection may consist of the mean square values of sample records obtained from different random signals, or sample records obtained at different times from the same random signal, or sample records obtained by dividing a single sample record into several subrecords. The practical applications represented by these various cases are discussed later in Section 16. 1. 5.

16. 1. 2 Review of Stationarity Test A

The following discussion is a review of Section 6. 1. 8 of Ref. [1]. Consider a stationary random signal $x(t)$ which hypothetically exists over all time. Let the mean value of $x(t)$ be zero, and the mean square value of $x(t)$ be σ_x^2. Assume a long sample amplitude time history record of length T is obtained which is then divided into N number of subrecords, each of equal length $T_1 = (T/N)$, as shown in Figure 16. 1.

For each of the subrecords, a mean square value $\overline{x_i^2}$ * $(i = 1, 2, 3, \ldots, N)$ is measured. Each value $\overline{x_i^2}$ will be an estimate of the true mean square value σ_x^2 for the signal. That is,

$$E\left[\overline{x_i^2}\right] = \sigma_x^2 \tag{16. 1}$$

* The mean square value of subrecords is denoted by s^2 in Section 6. 1. 8 of Ref. [1]. The term $\overline{x^2}$ is used here to avoid confusion with the experimental sample variance denoted by s^2 later in this section. Furthermore, a number of subscripts are used here for additional clarity which were not used in Section 6. 1. 8 of Ref. [1].

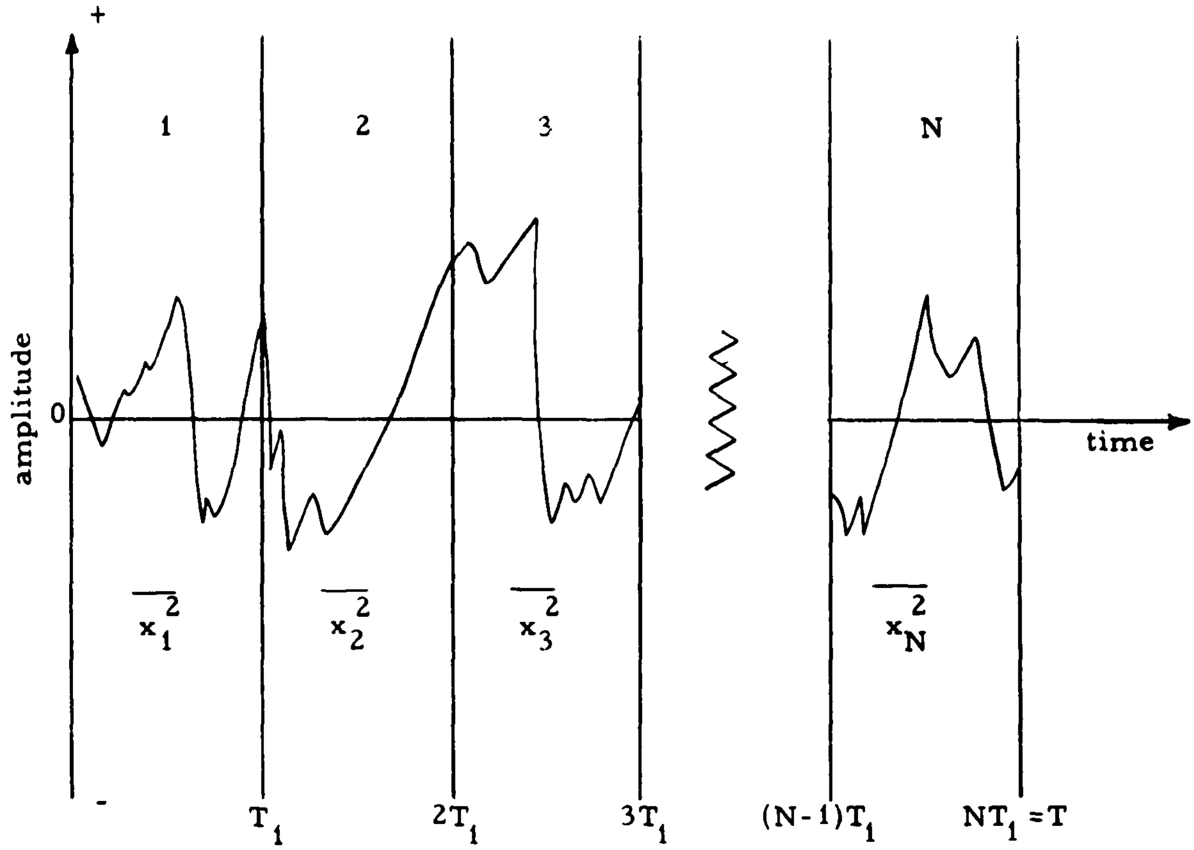

Figure 16. 1 Sample Amplitude Time History Record

Assuming $x(t)$ is distributed in an approximately Gaussian manner, the sampling distribution for the values $\overline{x_i^2}$ will be associated with a chi-squared distribution as follows.

$$\frac{\overline{x_i^2}}{\sigma_x^2} \sim \frac{\chi^2}{(n-1)} \quad ; \quad n = 2BT_1 \qquad (16.2)$$

where "$\sim$" means "distributed as", and χ^2 is a chi-squared distribution with (n - 1) degrees of freedom. The term n is the equivalent number of events represented by each subrecord. The bandwidth B is an equivalent ideal bandwidth with infinitely sharp cutoffs.

From Eq. (16.2), a probability statement concerning the expected values for $\overline{x_i^2}$ can be made as follows.

$$\text{Prob}\left[\frac{\sigma_x^2 \chi^2_{(1-\alpha/2)}}{(n-1)} \leq \overline{x_i^2} \leq \frac{\sigma_x^2 \chi^2_{(\alpha/2)}}{(n-1)}\right] = (1-\alpha) = p^*$$

The meaning of Eq. (16.3) is as follows. If $x(t)$ is stationary over the sample record of length T, the probability is p that the value $\overline{x_i^2}$ measured from each of the N number of subrecords will fall within the noted interval, or the probability is $(1-p)$ that the value will fall outside the noted interval.

Now, consider each measurement $\overline{x_i^2}$ as an experiment where N number of experiments are performed. Each experiment has two possible outcomes: the value $\overline{x_i^2}$ will fall outside the p probability interval (a failure), or the value $\overline{x_i^2}$ will fall inside the p probability interval (a success). Assuming the series of N number of experiments are statistically independent, the set of experiments may be considered as a set of Bernoulli trials where the probability of a success is p and the probability of a failure is $(1-p)$.

Let q be the number of failures. The sampling distribution for the number of failures in a finite number of experiments, N, is given by a binomial distribution as follows:

$$b(q) = \binom{N}{q}(1-p)^q (p)^{N-q} \tag{16.4}$$

where

$$\binom{N}{q} = \frac{N!}{q!\,(N-q)!}$$

From Eq. (16.4), if the signal $x(t)$ is stationary, the probability of actually obtaining up to k number of failures (values of $\overline{x_i^2}$ which fall outside the desired probability interval) is given by

$$\text{Prob}\,(q \leq k) = \sum_{q=1}^{k} \binom{N}{q}(1-p)^q (p)^{N-q} \tag{16.5}$$

*The term p is used here instead of $(1-\alpha)$ to avoid confusion with the usual concept of $(1-\alpha)$ = confidence coefficient.

The mean and variance for the binomial distribution in Eq. (16.4) is given by

$$\begin{aligned} \mu_q &= (1 - p)N \\ \sigma_q^2 &= (1 - p)pN \end{aligned} \tag{16.6}$$

For those cases where $(1 - p)N$ is relatively large, say greater than 10, the binomial distribution may be approximated by a normal distribution with a mean and variance as given in Eq. (16.6).

From either Eq. (16.5) or Eq. (16.6), a hypothesis test for stationarity can be performed at any desired level of significance. Further details are presented in Section 6.1.8 of Ref. [1].

<u>Example</u>

The above discussions will be clarified by considering a specific example. The example used is an actual case which is studied herein experimentally. Let the length of the sample record be T = 25 seconds. Let the number of subrecords formed from the sample record be N = 25. Suppose the effective bandwidth of the signal is B = 56 cps. Let the true mean square value of the signal, σ_x^2, be arbitrary. For this example,

$$T_1 = (T/N) = (25/25) = 1 \text{ second}$$

$$n = 2BT_1 = 2(56) = 112$$

and Eq. (16.3) becomes

$$\text{Prob}\left[\frac{\chi^2_{(1-\alpha/2)}\sigma_x^2}{111} \leq \overline{x_i^2} \leq \frac{\chi^2_{\alpha/2}\sigma_x^2}{111}\right] = p \tag{16.7}$$

Let p = (1 - p) = 0.5; that is, a 50% probability. From Table A-6b of Ref. [2] (noting that "percentiles" in this table are 1 - p),

$$\chi^2_{(0.75)} = 0.905$$

$$\chi^2_{(0.25)} = 1.09$$

Then, Eq. (16.7) reduces to

$$\text{Prob}\left[0.905\sigma_x^2 \leq \overline{x_i^2} \leq 1.09\sigma_x^2\right] = 0.50 \tag{16.8}$$

From Eq. (16.8), if the signal x(t) is stationary over the sample record, the probability is $p = 0.50$ that a value $\overline{x_i^2}$ will fall inside the stated limits (a success), and $(1 - p) = 0.50$ that a value $\overline{x_i^2}$ will fall outside the stated limits (a failure).

Now consider the sampling distribution for the number of failures q. From Eq. (16.6), the mean and variance are given by

$$\begin{aligned} \mu_q &= (1 - p)N = 12.5 \\ \sigma_q^2 &= (1 - p)pN = 6.25 \end{aligned} \tag{16.9}$$

Thus, for the 25 values for $\overline{x_i^2}$ ($i = 1$ to 25), the expected number of failures (values less than $0.905\sigma_x^2$ or greater than $1.09\sigma_x^2$) is 12.5. The sampling variance is 6.25. If the signal x(t) were not stationary over the record length T = 25 seconds, the expected number of failures would obviously be greater. Thus, to test a hypothesis of stationarity, a one-sided test is used.

For the present example, let it be hypothesized that x(t) is stationary over the record of length T = 25 seconds. This hypothesis will now be tested at the 5% level of significance. From Table A-4 of Ref. [2], for an $\alpha = 0.05$ level of significance, and a normal distribution for q, the following probability exists

$$\text{Prob}\left[q \leq (\mu_q + 1.65\sigma_q)\right] = 0.95 \tag{16.10}$$

Thus, the region of acceptance for the hypothesis test is

$$q \leq 16.625, \text{ or } q < 17 \tag{16.11}$$

For the 25 mean square estimates, $\overline{x_i^2}$, if 16 or fewer fall within the interval ($0.905\sigma_x^2$, $1.09\sigma_x^2$), the hypothesis of stationarity is accepted. If 17 or more fall outside the noted interval, the hypothesis

of stationarity is rejected with a Type I error of 5%. That is, the signal $x(t)$ is considered nonstationary over the record length of $T = 25$ seconds with a probability of 5% that this conclusion is an error.

Since the possible conditions of nonstationarity are undefined, a quantitative Type II error for the test is not really meaningful. However, in qualitative terms, when the test is applied at any given level of significance to a nonstationary signal, the Type II error will go down as the values of n and/or N are increased.

The application of this test for stationarity (Test A) to actual vibration data poses one major problem. The true mean square value of the vibration response signal σ_x^2 will be an unknown quantity. It is thus necessary in practice to replace σ_x^2 with the best available estimate. The best available estimate is obviously the mean square value $\overline{x^2}$ for the entire record of length T. That is,

$$\sigma_x^2 \approx \overline{x^2} = \frac{1}{T}\int_0^T x^2(t)\,dt = \frac{1}{N}\sum_{i=1}^{N} \overline{x_i^2} \qquad (16.12)$$

It is shown in Section 13. 5. 3 that Eq. (16. 12) should yield acceptable results if $2BT > 40N\,(2BT_1 > 40)$.

16. 1. 3 Principles of Stationarity Test B

The second test for stationarity to be discussed here is somewhat simpler in concept than Test A discussed in Section 16. 1. 2. As before, consider a stationary random signal $x(t)$ with a mean value of zero and a mean square value of σ_x^2. Assume a sample amplitude time history record of length T is obtained and divided into N number of subrecords, each of equal length $T_1 = (T/N)$ as shown in Figure 16. 1. For each of the subrecords, a mean square value $\overline{x_i^2}$ $(i = 1, 2, 3, \ldots, N)$ is measured. Each value of $\overline{x_i^2}$ will be an estimate of σ_x^2 as given by Eq. (16. 1). Assuming $x(t)$ is distributed in an approximately Gaussian manner, the sampling distribution for the values $\overline{x_i^2}$ will be as given by Eq. (16. 2).

It is proved in Section 13.5.2 that for stationary records the normalized variance of the mean square values $\overline{x_i^2}$ can be described theoretically as follows.

$$\epsilon^2 = \frac{\mathrm{Var}(\overline{x_i^2})}{(\sigma_x^2)^2} \approx \frac{1}{BT_1} \tag{16.13}$$

Given the set of N number of values $\overline{x_i^2}$, a variance for the values can be estimated by calculating the sample variance s^2 as follows.

$$s^2 = \frac{1}{N}\sum_{i=1}^{N}(\overline{x_i^2} - \overline{x^2})^2 = \frac{1}{N}\sum_{i=1}^{N}(\overline{x_i^2})^2 - (\overline{x^2})^2 \quad * \tag{16.14}$$

Here, $\overline{x^2}$ is the average mean square value given by Eq. (16.12). From Eq. (16.14), an estimate for the normalized variance of the values $\overline{x_i^2}$ will be

$$\hat{\epsilon}^2 = \frac{s^2}{(\sigma_x^2)^2} \tag{16.15}$$

The term $\hat{\epsilon}^2$ is simply a variance estimate and will thus have a sampling distribution associated with a chi-squared distribution as follows.

$$\frac{\hat{\epsilon}^2}{\epsilon^2} \sim \frac{\chi^2}{N} \tag{16.16}$$

where "$\sim$" means "distributed as" and χ^2 is chi-squared with (N - 1) degrees of freedom. The practical validity of Eq. (16.16) is experimentally substantiated in Section 13.

In Eq. (16.16), if the signal x(t) is stationary, the measured variance $\hat{\epsilon}^2$ and the theoretical variance ϵ^2 should be equivalent. Then, a hypothesis of stationarity is established by the null hypothesis H_o as follows.

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* A biased expression for s^2 is employed here so that all statistical procedures to follow will be consistent with procedures outlined in Ref. [1].

$$H_o : \hat{\epsilon}^2 = \epsilon^2 \tag{16.17}$$

If x(t) is nonstationary over the record of length T, the variance of the values $\overline{x_i^2}$ would be greater than predicted by ϵ^2. Thus, a one-sided test of H_o is used. More specifically, if H_o is tested at the α level of significance, the region of acceptance is given by

$$\frac{\hat{\epsilon}^2}{\epsilon^2} < \frac{\chi_\alpha^2}{N} \tag{16.18}$$

A plot of the acceptance regions of the normalized variance ratio for various values of α and N is shown in Figure 16.2.

For clarity, consider the above discussion in terms of the same example used in Section 16.1.1. That is,

$$T = 25 \text{ seconds}$$
$$N = 25$$
$$B = 56 \text{ cps}$$
$$T_1 = 1 \text{ second}$$
$$\epsilon^2 \approx \frac{1}{BT_1} = 0.0178$$

Let it be hypothesized that x(t) is stationary over the record of length T = 25 seconds; that is, $H_o : \hat{\epsilon}^2 = \epsilon^2$. This null hypothesis will now be tested at the 5% level of significance. From Table A-6a of Ref. [2], $\chi_{0.05}^2 = 36.4$ for nine degrees of freedom, and the following probability exists.

$$P\left[\hat{\epsilon}^2 < \frac{(36.4)(0.0178)}{25}\right] = P\left[\hat{\epsilon}^2 < 0.0259\right] = 0.95 \tag{16.19}$$

Then the region of acceptance for the null hypothesis H_o is

$$\hat{\epsilon}^2 < 0.0259 \tag{16.20}$$

where $\hat{\epsilon}^2$ is computed using Eqs. (16.14) and (16.15). If the computed value for $\hat{\epsilon}^2$ is less than 0.0259, the hypothesis of stationarity is

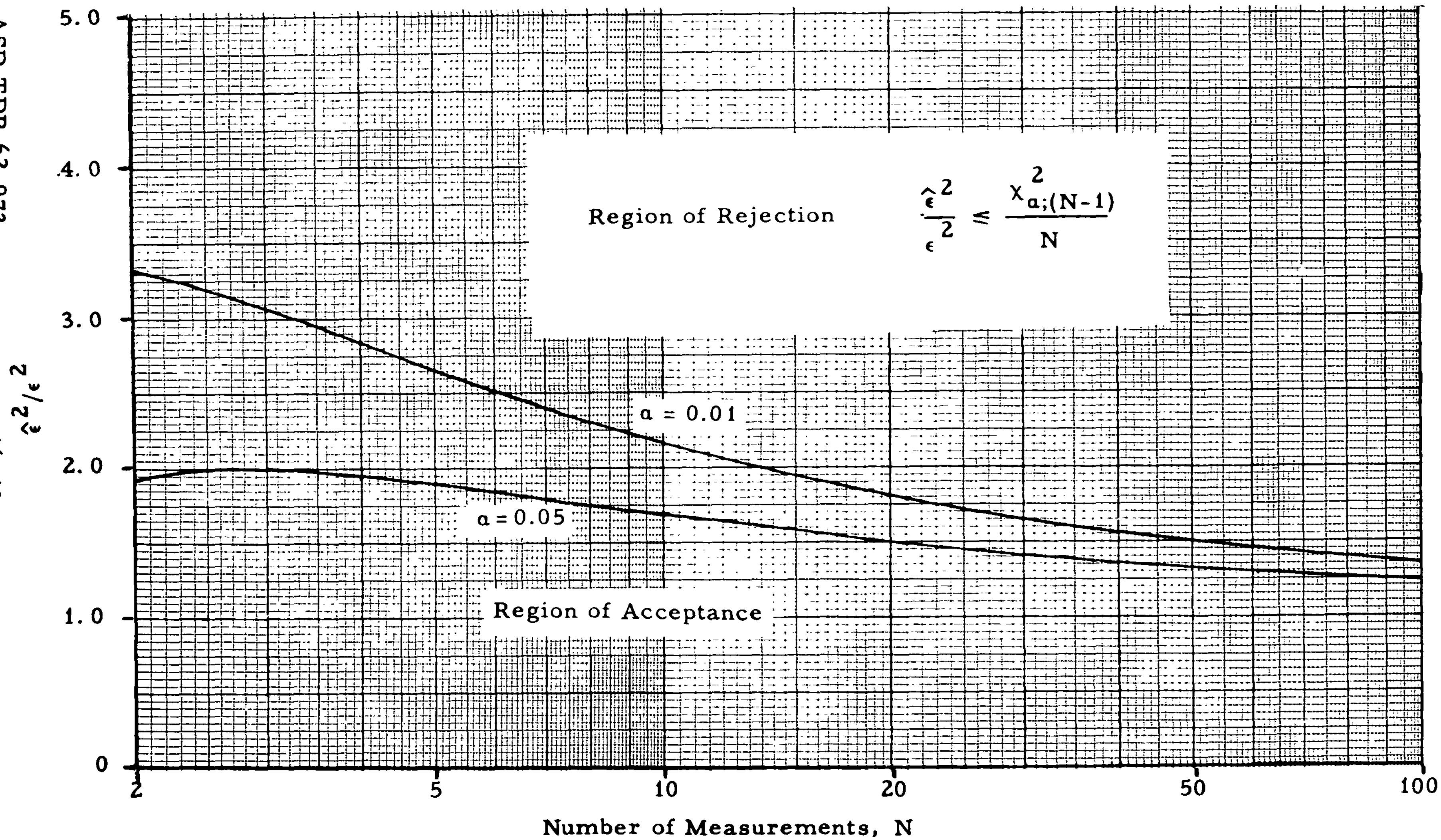

Figure 16.2 Acceptance Regions for Stationarity Test B

accepted. If $\hat{\epsilon}^2$ is greater than 0.0259, the hypothesis of stationarity is rejected with a Type I error probability of 5%. That is, the signal x(t) is considered nonstationary over the record length of T = 25 seconds with a probability of 5% that this decision is an error.

As in Test A, there is no meaningful Type II error probability, β, because there is no quantitative definition for all of the possible conditions of nonstationarity. For this test, however, a fictitious Type II error can be defined in terms of deviations in the ratio $E\left[\hat{\epsilon}^2\right]/\epsilon^2$. The Operating Characteristic (OC) curves for this case are given for $\alpha = 0.05$ and 0.01 in Figures 6.17 and 6.18 of Ref. [2]. The number of records N needed to detect a nonstationarity as measured by $E\left[\hat{\epsilon}^2\right]$ with a probability β may be selected from the appropriate OC curves. In general, the Type II error will go down as the value of N is increased.

The application of this test for stationarity (Test B) to actual vibration data poses the same problem discussed in Section 6.1.1 for Test A. The true mean square level of the vibration response signal σ_x^2, needed to compute $\hat{\epsilon}^2$ from Eq. (16.15), will be an unknown quantity. For practical applications it is necessary to replace σ_x^2 by the mean square value $\overline{x^2}$ computed for the entire record of length T, as shown in Eq. (16.12). This substitution is acceptable if $2BT_s \gg 40$.

16.1.4 Principles of Stationarity Test C

The third test for stationarity to be discussed here is simplest in terms of practical applications of the three procedures presented. As before, consider a stationary random signal x(t) with a mean value of zero and a mean square value of σ_x^2. Assume a sample amplitude time history record of length T is obtained and divided into N number of subrecords, each of equal length $T_1 = (T/N)$, as shown in Figure 16.1. For each of the subrecords, a mean square value $\overline{x_i^2}$ $(i = 1, 2, 3, \ldots, N)$ is measured. Each value $\overline{x_i^2}$ will be an estimate of σ_x^2 based on $n = 2BT_1$ equivalent number of events.

Now, select from the N number of values $\overline{x_i^2}$ the maximum value $\left(\overline{x^2}\right)_{max}$ and the minimum value $\left(\overline{x^2}\right)_{min}$. Assuming x(t) is distributed

in an approximately Gaussian manner, the ratio of these two values will be distributed as the statistic F_{max}. That is,

$$\frac{(\overline{x^2})_{max}}{(\overline{x^2})_{min}} \sim F_{max} \tag{16.21}$$

where "$\sim$" means "distributed as" and F_{max} is an F_{max} distribution for N number of mean square estimates, each with $(n - 1)$ degrees of freedom.

In Eq. (16.21), if the signal $x(t)$ is stationary, the measured value $(\overline{x^2})_{max}$ and $(\overline{x^2})_{min}$ should be equivalent. Then, a hypothesis of stationarity is established by the null hypothesis H_o as follows

$$H_o : (\overline{x^2})_{max} = (\overline{x^2})_{min} \tag{16.22}$$

If H_o is tested at the α level of significance, the region of acceptance is given by

$$\frac{(\overline{x^2})_{max}}{(\overline{x^2})_{min}} < F_{max\,;\,\alpha} \tag{16.23}$$

where F_{max} for $\alpha = 0.01$ and 0.05 are tabulated in Table 5.7 of Ref. [1]. A plot of the acceptance regions of the max-min mean square ratio for various values of n and N is shown in Figure 16.3.

The obvious advantage of Test C is its simplicity of application. Only one division of the maximum and minimum mean square estimates is required. There are two possible disadvantages to the test. First, the F_{max} statistic has not been studied in great detail and only limited tables of $F_{max\,;\,\alpha}$ values are available. Second, the power of the test is probably less than for other procedures which employ all available mean square estimates to make decisions. The latter problem will be considered in some detail by the experiments performed herein.

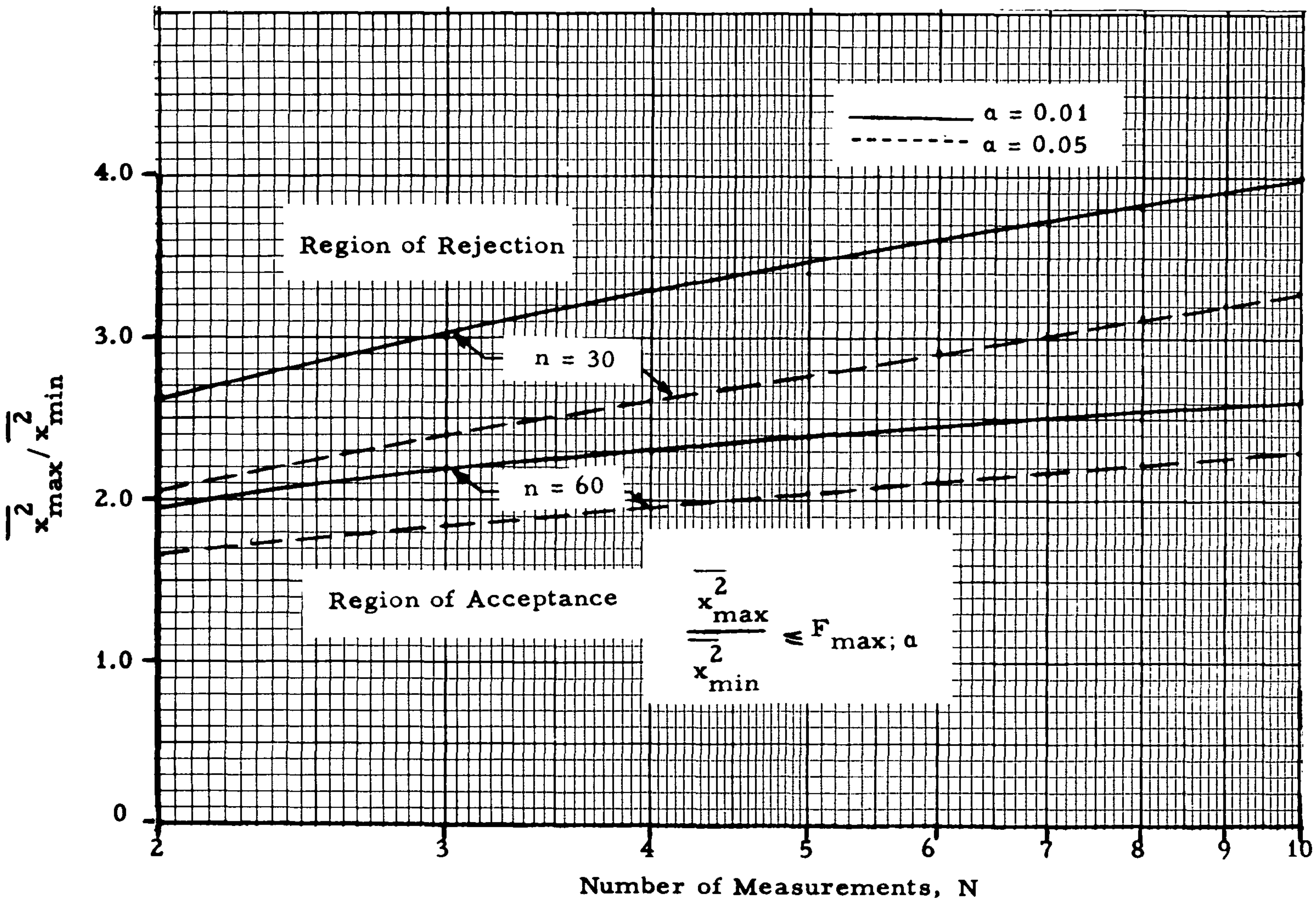

Figure 16.3 Acceptance Regions for Stationarity Test C

16.1.5 Bandwidth and Averaging Time Considerations

Each of the three tests for stationarity proposed here is applicable to any collection of mean square estimates which one wishes to test for equivalence. However, for all three tests, it is required that the mean square estimates each represent an equal number of events n, and that the value of n is known. From Eq. (16.2), n is a function of the signal bandwidth B and the record length (averaging time) T_1 for each record of the collection. Bandwidth and averaging time considerations are discussed in Section 13.1.3. However, some additional discussion is warranted here.

a) Bandwidth Considerations

If a random signal $x(t)$ with a uniform power spectrum, $G(f)$ = constant, is filtered with infinitely sharp cutoffs between any two frequencies f_a and f_b, the bandwidth of the signal, $B = f_b - f_a$, is ideally defined. Given a real filter with a constant parameter linear frequency response function $H(f)$, an equivalent ideal bandwidth for the filter is given by

$$B = B_N = \frac{1}{(H_{max})^2} \int_{-\infty}^{\infty} |H(f)|^2 \, df \qquad (16.24)$$

For stationary signals, the power spectral density function for the response, $G_x(f)$, is associated with the power spectral density function for the excitation, $G_e(f)$, as follows.

$$G_x(f) = |H(f)|^2 \, G_e(f) \qquad (16.25)$$

For most practical purposes, the power spectrum for a vibration response may be considered the result of linear filtering of an excitation with a uniform power spectrum, $G_e(f)$ = constant. Then, upon substitution of Eq. (16.25) into Eq. (16.24), the following result is obtained.

$$B = \frac{1}{G_{x\,max}} \int_{-\infty}^{\infty} G_x(f) \, df = \frac{\sigma_x^2}{G_{x\,max}} \qquad (16.26)$$

In words, Eq. (16. 26) states that the equivalent ideal bandwidth for a random signal representing a vibration response $x(t)$, is given by the mean square value divided by the peak value of the power spectrum. Thus, given the estimates $\overline{x^2}$ for the mean square value and $\hat{G}(f)$ for the power spectrum, the equivalent ideal bandwidth for a random signal can be estimated from the ratio $\overline{x^2}/\hat{G}_{max}$.

b) <u>Averaging Considerations</u>

The ideal method of averaging is by linear integration over the record length T. For the problem at hand, the mean square values of the subrecords should be determined by linear integration over each of the record lengths T_1. However, averaging is often accomplished by continuous smoothing with a low pass RC filter having a time constant K. The general associations between the time constant K and the equivalent averaging time T_1 are discussed in Section 13. 1. 3. There are special considerations which apply to the present problem.

Assume a long record of length T is to be tested for stationarity by analysis of a continuous RC averaged mean square estimate as shown in Figure 16. 4.

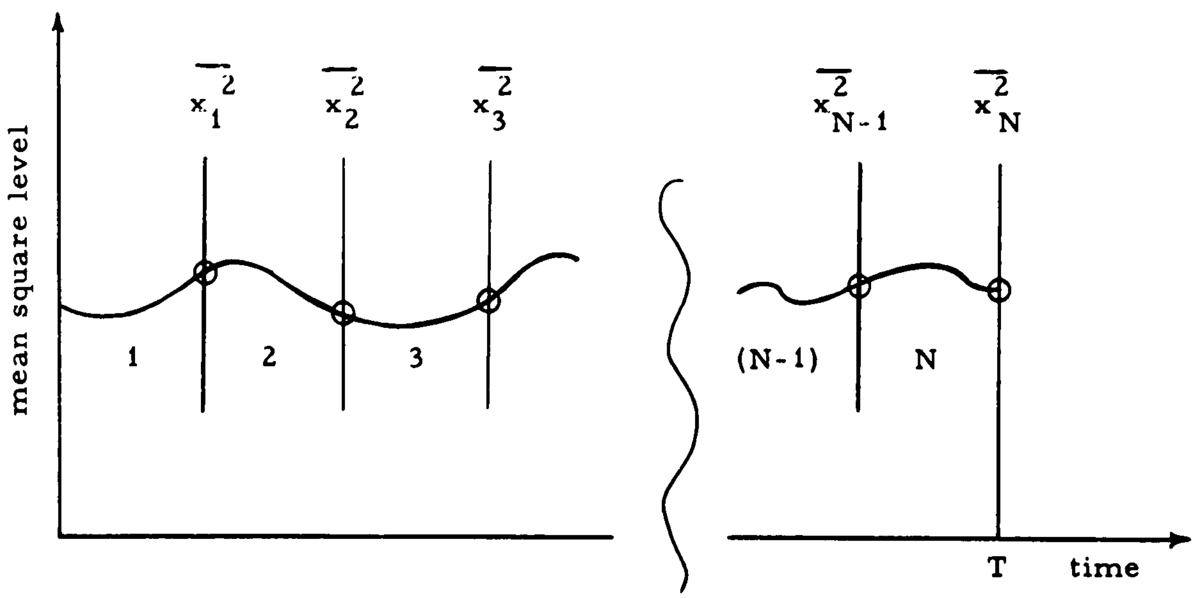

Figure 16. 4 Analysis of Continuous Mean Square Levels

Let the time constant K be short as compared to T; that is, $K \ll T$. Now divide the continuous mean square estimate into N equal intervals such that each interval is about 3K or 4K long. The level of the continuous mean square estimate at the end of each interval will constitute a mean square measurement $\overline{x_i^2}$ based on an equivalent averaging time of $T_1 = 2K$. The individual readings should be 3K or 4K apart to assure that they are statistically independent. Note that the points at which readings are made can also be selected by random sampling. Under no circumstances, however, should the points be selected without a pre-designated plan (equal intervals or random intervals). Otherwise, the person reducing the data may bias the results.

16.2 DESIGN OF EXPERIMENTS AND PROCEDURES

16.2.1 General Design and Procedures

The general purpose of these experiments is twofold. The first objective is to experimentally verify the theoretical foundation for Stationarity Test A. The foundation for Stationarity Tests B and C is based directly on the theoretically expected variability of mean square estimates, which is studied and substantiated by experiments in Section 13. The second objective is to empirically compare the effectiveness or power of the three tests by applying them to data obtained from a signal record with a known nonstationary characteristic.

16.2.2 Procedures for Verifying Stationarity Test A

The objective is to confirm the validity of Eq. (16.4), which is the basis for establishing an acceptance region for the test of a stationary hypothesis. Given a collection of N mean square estimates and a χ^2 probability interval of p probability, Eq. (16.4) defines the sampling distribution for the number $\hat{q}$ of the N estimates which will fall outside the interval (the number of failures). The sampling distribution is ideally a binomial distribution with a mean value and variance as follows.

$$\mu_q = (1 - p) N \qquad (16.27a)$$

$$\sigma_q^2 = (1 - p) pN \qquad (16.27b)$$

The verification procedure is to apply the Stationarity Test A to a collection of N mean square estimates from a stationary random signal, and determine the number of failures q. The experiment is repeated M number of times to obtain a collection of M number of experimental values for q. A sample mean and variance for q are computed as follows.

$$\bar{q} = \frac{1}{M} \sum_{i=1}^{M} q_i \tag{16.28a}$$

$$s^2 = \frac{1}{M} \sum_{i=1}^{M} (q_i - \bar{q})^2 = \frac{1}{M} \sum_{i=1}^{M} q_i^2 - \bar{q}^2 \tag{16.28b}$$

The expected values for the above sample mean and variance are as follows.

$$E\left[\bar{q}\right] = \mu_q \tag{16.29a}$$

$$E\left[s^2\right] = \left(\frac{M - 1}{M}\right) \sigma_q^2 \tag{16.29b}$$

The sample mean $\bar{q}$ will be distributed as follows.

$$(\bar{q} - \mu_q) \sim \frac{s\,\text{"t"}}{M - 1} \tag{16.30a}$$

where "t" is a student's "t" distribution with (M - 1) degrees of freedom. The sample variance s^2 will be distributed as follows.

$$\frac{s^2}{\sigma_q^2} \sim \frac{\chi^2}{M} \tag{16.30b}$$

where χ^2 has (M - 1) degrees of freedom. Equations (16.30a) and (16.30b) permit the experimentally determined values for μ_q and σ_q^2, as given by $\bar{q}$ and s^2, to be tested for equivalence to the theoretical values for μ_q and σ_q^2, as given by Eq. (16.27). The existence of equivalence will be considered as confirmation of the general form of Eq. (16.4) and Stationarity Test Procedure A.

More specifically, the Stationarity Test A is applied to a random signal with a bandwidth of B = 56 cps on a center frequency of f_c = 1000 cps. A total of N = 25 mean square estimates are gathered by true averaging (linear integration) over consecutive subrecords, each of length T_1 = 1 second. A χ^2 probability interval of p = 0.50 is employed, giving the limits for a success or failure as stated in Eq. (16.8). These conditions are consistent with the numerical example given in Section 16.1.2. The experiment is repeated M = 31 times. Equations (16.27) and (16.28) become

$$\mu_q = (1 - 0.5)\,25 = 12.5$$

$$\sigma_q^2 = (1 - 0.5)\,0.5(25) = 6.25$$

$$\bar{q} = \frac{1}{61}\sum_{i=1}^{61} q_i$$

$$s^2 = \frac{1}{61}\sum_{i=1}^{61} q_i^2 - \bar{q}^2$$

The hypothesis of equivalence between the theoretical and experimental values is tested at the α = 0.10 level of significance. Then, from Eqs. (16.30), the region of acceptance for the hypothesis of equivalence is

$$\frac{-1.70\,s}{30} < (\bar{q} - \mu_q) < \frac{1.70\,s}{30} \qquad (16.31a)$$

$$0.596 < \frac{s^2}{\sigma_q^2} < 1.41 \qquad (16.31b)$$

If the 31 experimentally determined values for q have a sample mean and variance which fall within the above limits, the hypothesis of equivalence will be accepted, and Eq. (16.4) will be considered valid. If the sample mean and variance fall outside the noted limits, the hypothesis of equivalence will be rejected, and there will be reason to question the validity of Eq. (16.4) and the entire procedure for Stationarity Test A.

The probable Type I and Type II errors for the experiment are $\alpha = \beta = 0.10$ for detecting a deviation of 12% in $\mu_q = 12.5$ and a deviation of about 2.4 to 1 in $\sigma_q^2 = 6.25$ (a deviation of approximately 50% in σ_q). These results follow directly from the procedures for determining a sample size for equivalence of means and variances, as presented in Section 12.2.2. For these experiments, the sample size of interest is the number of experiments M. Thus, from Eq. (12.19), the relationship between the sample size M, the theoretical standard deviation σ_q, and the mean value deviation $\Delta\mu_q$ which is to be detected for $\alpha = \beta = 0.10$ is given by

$$M = 10.8 \left(\frac{\sigma_q}{\Delta\mu_q} \right)^2 \approx 31 \tag{16.32}$$

Hence, for $\sigma_q = 2.5$, the difference in means is $\Delta\mu_q = 1.5$, which is 12% of $\mu_q = 12.5$.

From Eq. (12.22), the relationship between M and the deviation $\Delta\sigma_q^2$ which is to be detected for $\alpha = \beta = 0.10$ is given by

$$\frac{\sigma_q^2 + \Delta\sigma_q^2}{\sigma_q^2} = \frac{\chi^2_{(0.05)}(M-1)}{\chi^2_{(0.95)}(M-1)} \approx 2.4 \tag{16.33}$$

for $M = 31$.

16.2.3 Generation of Nonstationary Data

To perform some of the experimental studies to follow, a laboratory signal with a defined condition of nonstationarity is required. The specific condition of nonstationarity used is as shown in Figure 16.5. This is obtained by merely changing the mean square voltage level in a random noise generator.

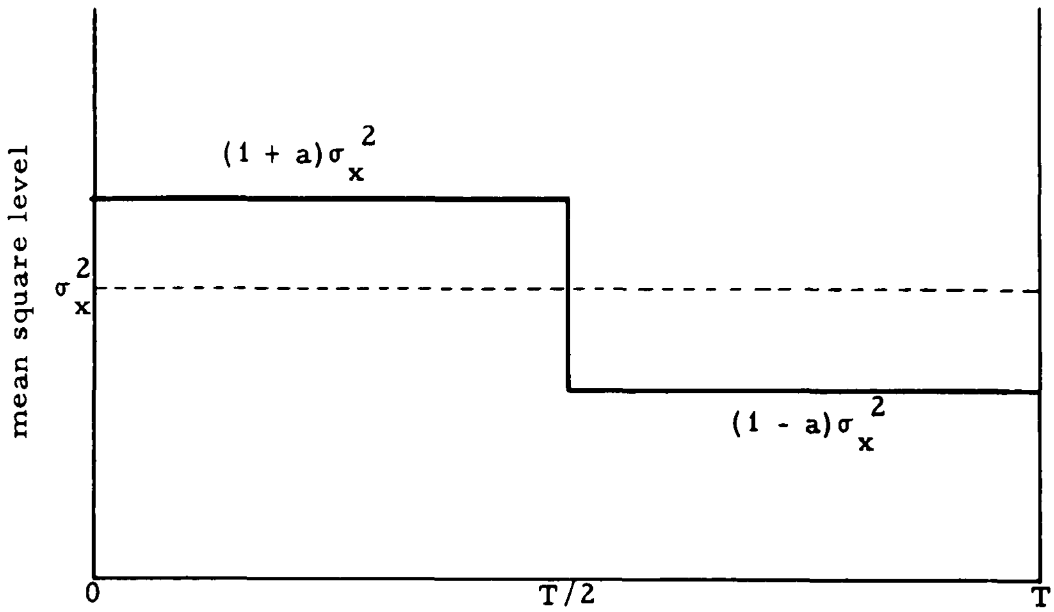

Figure 16.5 Condition of Nonstationary

In Figure 16.5, the parameter a is a fractional deviation in the actual mean square value of the signal from some average mean square level σ_x^2. For example, if $a = 0.10$, the nonstationarity is represented by a signal with a mean square level that is 10% above the average level σ_x^2 for the first half of the record and 10% below the average level σ_x^2 for the second half of the record. The magnitude of nonstationarity is measured in terms of the deviation a.

16.2.4 Procedures for Comparing Stationarity Tests A, B, and C

The objective is to empirically study the relative effectiveness of Stationarity Tests B and C, along with Test A, assuming its validity is substantiated, when they are applied with the same level of significance to the same nonstationary data. A nonstationary signal is contrived with the characteristics discussed in Section 16.2.3. That is, a sample record of length T is obtained from a signal whose true mean square value is some $\pm a$ % above and below an average level for each half of the record. A collection of N number of subrecords are gathered and tested for stationarity by each of the three tests at an α level of significance. The experiment is repeated M number of times to

empirically establish how often each test will accept or reject a stationary hypothesis. The group of experiments is performed for k number of different cases of nonstationarity.

More specifically, Stationarity Tests A, B, and C are applied to a random signal with a bandwidth of $B = 56$ cps on a center frequency of $f_c = 1000$ cps. A total of $N = 10$ mean square estimates are measured at equally spaced intervals from a continuous RC averaged mean square level time history plot. The continuous mean square estimate is for an averaging time constant of $K \approx 0.27$ seconds giving an effective averaging time per subrecord of $T_1 = 2K \approx 0.54$ seconds. Thus, each estimate is based on $n = 2BT_1 \approx 60$ equivalent number of events. The interval between the individual estimates is one second (about 4K) to assure statistical independence. For an actual example of a continuous RC averaged mean square estimate, see Figure 13.3. The ten mean square values are tested for stationarity by each of the three tests at the $\alpha = 0.05$ level of significance.

The experiment is repeated $M = 31$ times for $k = 3$ different cases as follows.

Case 1. All ten mean square estimates are obtained from a stationary random signal.

Case 2. The mean square estimates are obtained from a random signal with a true mean square value of $1.1\sigma_x^2$ for the first five estimates and $0.9\sigma_x^2$ for the last five estimates where σ_x^2 is an arbitrary average mean square level for the record.

Case 3. The mean square estimates are obtained from a random signal with a true mean square value of $1.2\sigma_x^2$ for the first five estimates and $0.8\sigma_x^2$ for the last five estimates.

For Case 1, each of the stationarity tests should accept the hypothesis of stationary at the 5% level of significance about 95% of the time. A rejection of the hypothesis is a Type I error which is fixed at 5% ($\alpha = 0.05$) for each test. Thus, for 31 experiments, one or two Type I errors would be anticipated. For Cases 2 and 3, each

of the stationarity tests would ideally reject the hypothesis of stationarity. An acceptance for these cases is a Type II error. For each test, an empirical probability for the Type II error (β), and a power factor for the test ($1 - \beta$) is computed as follows.

$$\beta = \frac{\text{number of acceptances}}{31}$$
$$\text{Power of Test} = 1 - \beta \tag{16.34}$$

These empirical quantities permit comparison of the relative power of the three stationarity tests.

16.2.5 Acceptance Intervals for Stationarity Tests

(a) Test A

From Eq. (16.3), for n = 60 and p = 0.50, the χ^2 probability interval is given by

$$0.870\,\overline{x^2} < \overline{x_i^2} < 1.12\,\overline{x^2} \tag{16.35}$$

where $\overline{x^2}$ is the average of the ten values for $\overline{x_i^2}$.

From Eq. (16.5), for N = 10, p = 0.50, and $\alpha = 0.05$, the region of acceptance for the hypothesis of stationarity is

$$q < 8 \tag{16.36}$$

The above number is arrived at by using binomial probabilities from Table A-29a of Ref. [2] and determining that value of k in Eq. (16.5) such that the probability is 0.95.

(b) Test B

From Eq. (16.18) or Figure 16.2, for N = 10 and $\alpha = 0.05$, the region of acceptance for the hypothesis of stationarity is

$$\frac{\hat{\epsilon}^2}{\epsilon^2} < 1.69 \tag{16.37}$$

Here, for n = 60 ($BT_1 = 30$), from Eq. (16.13), $\epsilon^2 = 0.0333$.

(c) Test C

From Eq. (16. 23) or Figure 16. 3, for n = 60, N = 10, and $\alpha = 0.05$, the region of acceptance for the hypothesis of stationarity is

$$\left(\frac{\overline{x^2_{max}}}{\overline{x^2_{min}}} \right) < 2.26 \tag{16.38}$$

16. 3 INSTRUMENTATION

The instruments and test set-up are exactly as employed for the experiments in Section 13. The power spectral density analyzer is used to obtain mean square estimates by either linear integration or RC averaging. See Section 13. 3 for details.

16. 4 RESULTS OF EXPERIMENTS

16. 4. 1 Verification of Stationarity Test A

The 25 mean square estimates gathered for each of the 31 experiments are presented in Table 16. 1. The average value for all of the 775 estimates collected is 5. 078. This value is used as the true mean square level σ_x^2 for the signal. Then, from Eq. (16. 3), the 50% χ^2 probability interval is

$$\text{Prob}\left[4.60 < \overline{x_i^2} < 5.54\right] = 0.50$$

The values presented in Table 16. 1 have two significant figures. The upper χ^2 limit poses no problem since a value of 5. 5 is definitely inside the interval and 5. 6 is definitely a failure q. However, for the lower limit of 4. 60, a problem arises. A value of 4. 6 means that the mean square estimate is between 4. 55 and 4. 65. It could be inside or outside the χ^2 interval. For these reasons, a value of 4. 6 in Table 16. 1 is counted as half a failure.

The number of failures q for each experiment in Table 16. 1 are presented in Table 16. 2.

Exper. Number	Mean Square Estimates $\overline{x_i^2}$ for Stationary Signal $B = 56$ cps, $f_c = 1000$ cps, $T_a = 1$ second (true averaging), $n = 112$																								
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1	4.6	4.7	5.6	5.0	5.6	4.5	5.5	6.0	5.8	4.3	4.8	4.3	4.2	6.1	6.1	5.0	3.9	5.1	5.0	5.5	4.4	5.1	6.0	5.5	4.8
2	5.5	5.2	5.4	5.4	5.7	4.7	6.3	4.8	5.0	6.1	5.0	4.7	5.3	5.3	5.7	4.6	6.0	5.8	4.7	5.9	6.8	4.7	6.2	4.3	3.3
3	5.9	5.0	6.9	5.9	5.5	4.9	5.1	5.1	5.1	4.4	3.6	5.8	5.1	6.1	6.1	5.5	4.3	4.9	5.6	5.2	4.4	4.5	6.2	6.2	5.8
4	5.5	5.2	7.6	4.7	5.3	5.3	5.0	5.5	4.9	5.1	5.8	6.6	4.5	4.5	4.9	5.3	4.5	4.1	5.4	4.7	7.5	5.0	5.0	5.5	6.3
5	5.9	4.9	5.0	5.4	5.9	5.8	5.1	4.9	5.5	5.1	5.4	4.9	6.5	6.2	5.0	4.4	4.8	5.2	5.3	5.0	4.3	4.9	5.3	4.4	4.3
6	5.2	4.4	4.8	4.8	3.9	4.4	5.8	5.8	4.4	4.5	5.2	5.4	3.9	5.4	4.2	4.4	5.9	4.6	5.3	5.1	4.5	3.9	5.0	5.1	4.2
7	4.9	5.4	5.3	5.4	3.8	4.2	5.2	5.3	4.5	5.4	7.7	6.6	5.6	4.0	3.4	3.4	4.4	4.3	5.3	4.4	4.7	5.5	4.7	4.8	6.1
8	5.7	5.1	6.1	6.6	4.9	3.9	5.1	5.5	5.4	4.1	4.7	6.8	6.1	4.8	4.9	4.2	5.2	5.7	5.9	5.2	5.7	5.6	4.9	4.0	4.3
9	5.0	5.6	5.9	5.0	5.8	5.4	5.4	5.0	4.6	3.9	6.0	5.7	5.1	5.5	5.9	4.6	4.3	4.8	4.6	5.6	5.2	5.2	5.4	4.3	4.0
10	4.7	5.2	5.3	5.3	5.5	4.1	4.0	4.9	5.1	5.3	5.6	5.1	6.3	3.8	4.9	5.0	5.1	5.5	4.7	4.6	5.5	5.3	5.4	5.1	5.2
11	4.6	4.7	4.9	5.8	3.5	5.1	5.1	4.1	3.6	4.4	5.4	6.6	5.5	5.5	5.7	5.7	4.0	5.2	4.1	5.9	5.9	5.7	5.2	4.6	5.3
12	5.0	6.2	4.3	5.5	3.6	6.9	6.3	5.4	5.3	6.0	4.9	4.5	5.7	5.0	4.2	5.2	6.0	6.0	5.5	4.8	5.8	5.4	4.9	5.1	5.9
13	5.5	4.6	5.4	5.3	3.7	5.6	6.2	3.7	4.6	5.0	5.9	4.7	4.4	5.8	4.7	6.0	5.2	4.5	5.1	4.3	4.3	5.7	4.3	4.6	4.9
14	5.8	5.3	5.8	6.0	5.7	5.0	5.8	4.6	5.3	4.0	4.3	4.7	4.0	4.4	5.8	5.0	6.3	5.0	5.1	5.2	4.9	5.3	4.9	4.4	5.5
15	6.1	5.2	5.5	5.9	4.7	6.0	4.3	4.8	4.9	5.2	4.6	4.9	6.0	4.4	5.5	5.1	5.2	5.0	4.4	5.3	5.0	6.3	4.2	4.2	5.3
16	5.3	5.7	5.0	5.4	4.9	5.4	4.3	6.2	5.5	4.8	5.0	5.5	5.0	5.4	5.6	6.3	4.9	4.7	5.8	5.3	4.9	5.1	4.6	6.5	6.3
17	5.1	4.3	5.2	4.3	6.4	4.9	4.0	4.7	4.8	4.6	7.0	5.7	4.2	4.2	6.0	5.0	4.8	5.5	5.2	4.2	4.0	4.9	5.6	5.8	5.5
18	5.4	6.0	5.2	5.1	5.3	5.3	5.5	5.5	5.2	5.0	5.6	4.3	4.6	5.5	4.6	4.0	4.2	4.9	4.8	5.2	4.3	4.9	5.7	6.6	4.5
19	4.6	5.1	6.3	6.4	4.2	6.0	4.8	5.0	4.7	4.4	4.8	5.8	6.8	3.4	5.3	5.5	4.3	4.9	5.0	5.7	5.5	4.6	4.6	5.1	4.3
20	4.0	4.1	7.3	5.1	5.1	5.1	3.8	4.3	5.6	3.8	6.0	4.2	6.4	4.1	5.0	4.6	5.0	4.7	5.2	4.4	5.1	5.5	5.2	5.5	4.2
21	4.5	5.3	6.3	4.7	4.6	5.8	4.9	4.5	4.6	5.0	4.5	5.3	3.7	5.5	5.2	4.6	4.2	4.4	7.1	4.5	5.4	6.5	6.1	5.1	4.5
22	6.5	5.5	4.5	5.0	5.7	4.7	5.3	5.7	4.2	4.8	4.9	5.1	5.1	4.1	5.0	4.9	4.3	5.8	4.4	4.4	5.0	4.7	4.6	6.2	5.1
23	6.2	4.8	3.8	4.7	4.5	5.2	5.0	6.1	5.1	4.4	5.9	4.5	4.5	5.2	4.6	5.2	4.8	4.3	5.5	5.7	4.6	5.9	5.0	7.5	5.3
24	5.3	3.9	5.3	4.4	5.5	4.5	4.0	5.4	5.5	3.4	5.1	4.7	5.9	4.6	5.7	4.1	4.9	6.0	5.1	5.0	5.1	5.1	5.1	5.7	5.3
25	5.9	4.6	5.2	5.0	5.3	4.3	5.1	6.1	5.3	4.5	4.9	5.1	4.7	5.1	3.7	5.3	5.7	4.4	4.6	5.0	5.6	4.5	5.5	5.2	5.2
26	4.2	4.5	5.9	4.9	6.4	6.3	4.7	5.5	5.1	3.9	4.8	6.0	4.0	4.7	5.0	6.0	5.0	4.6	5.6	4.7	3.6	3.4	4.9	5.6	4.8
27	5.0	4.9	4.2	4.4	4.0	4.3	3.8	5.1	4.6	5.1	6.7	3.7	4.5	5.4	4.6	4.2	4.6	5.6	3.9	5.2	4.5	4.7	4.9	4.6	4.3
28	3.9	5.2	4.3	5.0	5.3	4.4	5.0	5.2	4.5	5.6	4.6	3.4	4.1	4.8	4.2	6.1	5.6	6.4	4.8	6.2	4.8	5.5	4.5	4.6	5.4
29	4.4	4.4	5.5	3.9	5.1	4.9	4.3	5.1	4.4	5.9	6.1	3.9	4.8	4.6	4.8	5.6	4.1	4.8	5.7	5.6	5.9	5.7	5.6	5.3	5.2
30	5.2	5.4	5.1	4.5	5.9	4.5	4.6	4.6	4.5	6.5	4.6	4.7	5.1	6.6	3.9	4.3	5.6	4.3	3.3	5.0	3.9	6.1	5.9	6.8	3.8
31	3.6	5.1	5.3	5.7	5.6	3.9	3.9	4.1	5.5	4.1	5.5	4.8	4.9	5.1	5.7	5.5	4.7	4.0	4.9	5.4	5.5	3.5	5.8	4.6	4.6

Table 16.1. Mean Square Estimates for a Stationary Random Signal (n = 112)

Number of Mean Square Estimates Outside 50% χ^2 Interval (Failures) for Each of 31 Experiments					
Exper. Number	q Failures	Exper. Number	q Failures	Exper. Number	q Failures
1	13.5	11	14	21	14.5
2	10.5	12	13	22	11.5
3	15	13	14.5	23	13
4	9	14	12.5	24	10.5
5	9	15	10.5	25	10
6	14.5	16	8.5	26	13.5
7	13	17	13.5	27	15
8	14	18	10	28	14
9	12.5	19	12.5	29	15.5
10	5.5	20	13.5	30	17.5
				31	12

Table 16.2 Number of Failures from Table 16.1

Using Eq. (16.28), the sample mean and variance for the 31 values of q in Table 16.2 are as follows.

$$\bar{q} = 12.45$$

$$s^2 = 6.0$$

$$s = 2.4$$

From Eqs. (16.30) and (16.31) where $s = 2.4$, the regions of acceptance for the hypotheses that $\bar{q} = \mu_q$ and $s^2 = \sigma_q^2$ are as follows.

$$-0.14 < (\bar{q} - \mu_q) < 0.14$$

$$0.60 < \frac{s^2}{\sigma_q^2} < 1.4$$

Since $\mu_q = 12.5$ and $\sigma_q^2 = 6.25$,

$$\bar{q} - \mu_q = 12.45 - 12.50 = -0.05$$

$$\frac{s^2}{\sigma_q^2} = \frac{6.0}{6.25} = 0.96$$

Thus, the hypothesis of equivalence for both the mean value and variance is accepted and Eq. (16.4) is considered valid.

16.4.2 Results for Stationarity Tests A, B, and C

The ten mean square estimates gathered for each of the 31 experiments with a stationary random signal (Case 1) are presented in Table 16.3. The results of the experiments for a ± 10% nonstationary random signal (Case 2) are presented in Table 16.4. The results of the experiments for a ± 20% nonstationary random signal (Case 3) are presented in Table 16.5.

Results for the application of Stationarity Test A to Cases 1, 2, and 3 are shown in Tables 16.6, 16.7, and 16.8, respectively. The results for Stationarity Test B are shown in Tables 16.9, 16.10, and 16.11. The results for Stationarity Test C are shown in Tables 16.12, 16.13, and 16.14.

16.4.3 Comparison of Stationarity Tests A, B, and C

From Tables 16.6, 16.9, and 16.12, it is seen that the application of each stationarity test to stationary data produced a single rejection for 31 repeated experiments. This rejection is a Type I error (the hypothesis is rejected when in fact it was true). For a hypothesis test at the 5% level of significance which is repeated 31 times, an average of 1.5 Type I errors would be anticipated, so the single rejections are consistent with expectations.

For the application of each stationarity test to nonstationary data, a rejection is desired and an acceptance constitutes a Type II error (the hypothesis is accepted when in fact it is false). These data from Tables 16.7, 16.8, 16.10, 16.11, 16.13, and 16.14 are summarized in Table 16.15.

It is clear from Table 16.15 that Stationary Test B is the most powerful of the three tests studied. Stationarity Test C ranks second in power. Stationarity Test A, the test originally proposed in Ref. [1], is the least powerful.

Exper. Number	Mean Square Estimates $\overline{x_i^2}$ for Stationary Signal - Case 1 B=56 cps, f_c= 1000 cps, T_1 = 0.54 (RC averaging), n=60									
	1	2	3	4	5	6	7	8	9	10
1	2.25	1.78	2.26	2.07	1.91	2.85	2.00	2.45	2.19	3.20
2	2.30	2.86	2.68	2.15	2.54	2.32	2.00	2.23	3.05	2.15
3	2.36	2.03	2.00	2.57	2.21	1.47	2.44	3.28	3.27	2.49
4	3.43	2.31	1.54	2.89	1.90	2.08	2.19	2.18	2.52	2.20
5	2.02	2.25	2.73	2.49	2.60	2.17	3.09	2.35	2.21	3.00
6	1.56	1.40	2.12	2.11	2.16	2.69	1.83	2.74	2.32	2.50
7	2.99	2.43	2.51	1.96	2.04	2.46	2.09	1.86	2.21	2.30
8	2.66	3.18	1.94	2.20	2.29	2.55	1.89	2.54	2.50	2.36
9	2.11	2.20	2.36	3.01	3.03	2.53	2.43	2.45	2.08	2.53
10	2.50	3.25	2.80	2.50	2.54	2.50	2.10	2.50	1.79	2.26
11	2.32	2.66	2.74	2.11	2.99	1.95	1.89	2.91	2.01	3.40
12	2.60	2.40	2.49	2.63	2.64	2.11	2.61	2.89	1.62	2.20
13	2.56	2.09	3.05	2.21	2.30	2.03	1.98	2.31	2.00	2.03
14	2.44	3.35	2.30	1.80	2.40	2.35	2.30	2.41	2.21	1.64
15	2.47	1.66	2.39	1.95	2.19	2.11	2.92	3.16	2.60	1.89
16	1.33	2.61	2.35	2.90	2.32	2.98	2.22	2.11	2.45	2.07
17	1.90	2.99	2.51	2.41	1.90	2.00	2.42	1.71	1.74	2.13
18	2.15	2.22	1.98	2.92	2.27	3.09	2.47	2.69	2.07	2.72
19	2.36	2.86	2.80	2.65	2.12	2.77	2.76	1.68	2.04	2.45
20	1.74	2.51	2.45	2.27	2.18	2.92	1.99	2.63	2.69	2.50
21	2.33	2.00	2.04	2.12	1.90	3.63	2.21	1.99	2.40	1.37
22	1.63	2.55	2.04	2.91	2.40	3.10	2.49	2.53	2.39	2.89
23	2.14	2.00	1.97	2.23	1.72	2.91	2.70	3.00	2.02	2.90
24	2.25	2.11	2.51	1.74	2.13	1.82	2.52	2.55	2.46	1.84
25	2.90	2.72	2.67	1.90	2.40	2.67	2.19	2.40	2.30	2.41
26	2.68	2.23	1.92	2.15	2.35	2.72	1.65	2.41	2.30	2.53
27	3.10	2.24	2.30	2.36	1.67	2.50	2.25	2.89	2.71	2.12
28	2.09	2.53	2.25	1.84	2.92	2.95	2.72	2.76	3.39	2.38
29	2.76	2.45	2.50	2.22	2.30	2.71	1.75	2.55	2.52	2.14
30	2.61	2.22	2.78	1.96	2.44	2.52	2.20	1.53	2.11	2.01
31	2.49	2.32	2.79	2.68	2.52	2.41	2.81	2.78	2.49	2.13

Table 16.3. Mean Square Estimates for Stationary Random Signal - Case 1

Exper. Number	Mean Square Estimates $\overline{x_i^2}$ for $\pm$10% Nonstationary Signal - Case 2 B=56 cps, f_c = 1000 cps, T_1 = 0.54 seconds (RC averaging), n=60									
	1	2	3	4	5	6	7	8	9	10
1	3.54	1.95	2.86	1.88	1.97	2.47	2.00	2.20	2.22	1.68
2	2.29	3.48	2.87	2.45	2.20	2.49	1.52	1.70	2.02	2.09
3	2.51	2.39	3.27	2.19	3.04	2.43	1.70	1.64	1.82	1.97
4	2.54	3.07	3.58	3.04	2.07	1.95	2.34	2.06	2.02	2.45
5	2.20	2.49	2.47	2.22	2.20	2.10	1.61	2.12	1.75	1.69
6	2.15	3.13	3.11	2.12	3.11	2.04	1.35	3.09	3.15	1.50
7	2.36	2.65	2.46	2.65	3.98	2.00	1.60	2.71	2.02	2.34
8	3.00	2.12	2.68	3.24	2.32	2.34	2.00	2.08	2.11	1.96
9	2.40	2.71	2.33	2.84	2.28	2.62	2.25	2.11	2.43	1.95
10	2.43	2.75	2.64	2.37	2.65	2.27	2.29	2.04	1.97	2.43
11	2.24	3.53	2.21	2.49	2.81	2.03	1.43	1.71	1.92	2.60
12	2.30	2.38	2.74	2.87	2.38	2.25	2.45	2.27	1.76	2.25
13	2.55	2.68	1.91	2.09	2.24	2.47	1.74	1.71	1.90	1.74
14	2.17	2.22	3.10	2.59	2.72	2.41	1.62	2.45	1.54	2.00
15	2.57	2.15	2.71	2.50	1.76	2.30	1.80	1.69	2.51	2.40
16	2.50	2.40	2.47	2.11	1.92	1.73	2.21	1.93	2.05	1.88
17	2.89	2.60	2.97	3.22	2.10	2.44	2.51	2.08	2.66	2.36
18	2.57	2.82	2.31	2.17	2.39	2.09	2.43	1.89	2.08	2.35
19	2.50	2.92	3.21	2.29	2.24	2.01	2.18	2.49	2.05	1.81
20	2.92	2.05	2.14	1.54	2.55	2.42	1.66	1.90	1.85	1.76
21	2.28	2.43	2.09	2.64	2.62	2.60	2.32	2.05	1.79	2.00
22	3.25	3.04	2.00	3.87	2.25	2.20	1.60	2.18	2.56	1.50
23	2.96	2.18	2.15	1.84	2.38	1.80	2.09	1.62	2.02	2.53
24	2.13	3.22	2.90	2.26	1.97	2.84	2.33	3.17	2.11	2.12
25	2.32	2.87	1.95	2.83	3.32	1.86	1.86	2.52	2.00	2.51
26	1.94	2.71	1.92	2.22	2.69	1.79	2.73	1.99	1.68	1.85
27	2.48	3.00	2.39	3.15	3.00	1.96	2.15	1.61	1.76	1.59
28	2.27	3.10	2.47	2.50	1.92	1.69	2.49	1.59	1.82	2.13
29	2.39	1.91	2.43	2.47	2.45	1.83	2.31	3.03	2.10	1.61
30	2.34	2.30	3.47	2.92	3.35	2.02	2.34	1.60	1.73	2.05
31	2.89	2.60	2.97	2.88	1.94	2.10	2.49	1.98	1.80	2.33

Table 16.4. Mean Square Estimates for Nonstationary Random Signal - Case 2

Exper. Number	Mean Square Estimates $\overline{x_i^2}$ for ±20% Nonstationary Signal - Case 3 B=56 cps, f_c = 1000 cps, T_1 = 0.54 seconds (RC averaging), n = 60									
	1	2	3	4	5	6	7	8	9	10
1	2.02	1.90	1.61	2.30	2.43	1.12	0.92	1.75	0.98	1.09
2	1.66	1.85	2.20	1.47	1.93	0.93	1.01	1.02	0.92	1.33
3	2.08	1.68	2.25	1.61	2.33	1.43	1.44	0.89	1.20	1.04
4	2.00	2.46	1.74	1.53	2.42	1.05	0.98	1.25	0.93	0.81
5	3.92	2.11	1.78	1.81	2.41	1.40	1.23	1.15	1.70	0.93
6	3.55	2.01	1.70	2.00	2.08	1.20	1.69	1.40	0.94	1.29
7	1.91	2.96	1.49	2.12	2.41	1.52	1.30	1.07	0.99	1.01
8	2.06	2.28	2.40	1.66	2.59	1.38	1.97	1.16	1.09	1.10
9	2.28	2.48	2.30	1.74	2.36	1.17	1.38	1.10	1.19	1.07
10	1.70	2.02	2.30	2.27	1.69	1.24	1.38	1.22	1.28	1.06
11	2.62	1.76	1.96	2.46	1.75	1.57	1.35	1.32	1.25	2.00
12	1.50	1.83	2.27	2.50	2.54	1.70	0.97	1.12	1.52	1.67
13	2.42	2.50	2.29	1.92	1.98	0.93	1.58	1.20	1.20	1.29
14	2.28	1.95	2.40	2.10	2.20	1.38	1.42	1.03	1.28	1.25
15	2.59	2.34	2.20	1.60	2.42	1.19	1.19	0.81	1.12	1.21
16	1.96	2.31	2.36	2.17	2.94	1.47	1.37	1.46	1.11	1.37
17	1.70	1.73	2.30	2.38	1.85	0.84	0.83	1.08	1.58	1.53
18	1.90	2.04	2.08	2.18	2.29	1.29	1.85	1.20	1.17	1.30
19	2.06	2.02	1.82	1.58	2.04	1.44	1.00	1.55	1.57	1.13
20	1.72	1.67	2.38	2.53	2.32	1.28	1.08	1.73	1.37	1.23
21	2.68	1.93	1.90	2.06	2.02	1.64	1.40	1.72	1.24	1.67
22	1.89	2.64	2.44	2.43	2.97	1.08	0.98	1.48	1.37	1.25
23	1.82	2.16	2.14	2.61	1.91	1.22	1.40	1.09	1.16	1.55
24	2.42	2.88	1.70	1.53	1.93	1.30	1.12	1.23	1.31	1.70
25	2.20	2.17	1.70	1.91	2.82	1.04	1.68	1.38	1.07	1.20
26	2.05	2.70	1.51	2.70	1.69	1.40	1.43	1.20	1.60	1.26
27	1.67	2.20	2.27	1.62	2.40	1.27	1.04	1.39	2.01	1.37
28	1.83	3.02	2.99	2.50	2.40	1.32	1.37	1.10	1.26	1.43
29	2.33	2.03	2.12	1.88	2.20	1.24	1.40	1.24	1.22	1.42
30	1.90	2.36	2.61	2.18	2.63	1.40	1.39	1.70	0.96	1.60
31	2.25	2.06	2.42	2.27	3.00	1.37	1.10	1.03	1.62	1.24

Table 16.5. Mean Square Estimates for Nonstationary Random Signal - Case 3

Exper. Number	Results for Stationarity Test A Applied to Stationary Data - Case 1, Table 16.3				
	$\overline{x^2} \approx \sigma_x^2$	50% χ^2 interval		Number of Failures q	Region of Acceptance $q < 8$
		lower limit $0.870\ \overline{x^2}$	upper limit $1.12\ \overline{x^2}$		
1	2.296	2.00	2.57	5	accepted
2	2.428	2.11	2.72	3	
3	2.413	2.10	2.70	5	
4	2.324	2.02	2.60	4	
5	2.491	2.17	2.79	4	
6	2.143	1.86	2.40	6	
7	2.285	1.99	2.56	3	
8	2.411	2.10	2.70	3	
9	2.473	2.15	2.77	4	
10	2.474	2.15	2.77	4	
11	2.498	2.17	2.80	7	
12	2.419	2.10	2.71	2	
13	2.256	1.96	2.53	2	
14	2.320	2.02	2.60	3	
15	2.334	2.03	2.61	5	
16	2.336	2.03	2.62	3	
17	2.170	1.89	2.43	4	
18	2.458	2.14	2.75	4	
19	2.449	2.13	2.74	7	
20	2.388	2.08	2.67	4	
21	2.199	1.91	2.46	3	
22	2.493	2.17	2.79	5	accepted
23	2.359	2.05	2.64	8	rejected
24	2.193	1.91	2.46	7	accepted
25	2.456	2.14	2.75	2	
26	2.294	2.00	2.57	4	
27	2.414	2.10	2.70	4	
28	2.583	2.25	2.89	4	
29	2.390	2.08	2.68	3	
30	2.238	1.95	2.51	4	
31	2.542	2.21	2.85	1	accepted

Table 16.6. Results for Stationarity Test A Applied to Case 1

Exper. Number	Results for Stationarity Test A Applied to $\pm$10% Nonstationary Data - Case 2, Table 16.4				
	$\overline{x^2} \approx \sigma_x^2$	50% χ^2 interval		Number of Failures q	Region of Acceptance $q < 8$
		lower limit $0.870\,\overline{x^2}$	upper limit $1.12\,\overline{x^2}$		
1	2.277	1.98	2.55	6	accepted
2	2.311	2.01	2.59	4	
3	2.296	2.00	2.57	6	
4	2.512	2.18	2.81	7	
5	2.085	1.81	2.34	5	accepted
6	2.475	2.15	2.77	10	rejected
7	2.477	2.15	2.77	4	accepted
8	2.385	2.07	2.67	5	
9	2.392	2.08	2.68	3	
10	2.384	2.07	2.67	3	
11	2.297	2.00	2.57	6	
12	2.365	2.06	2.65	3	
13	2.103	1.83	2.36	6	
14	2.282	1.98	2.56	5	
15	2.239	1.95	2.51	6	
16	2.120	1.84	2.37	4	
17	2.583	2.25	2.89	5	
18	2.310	2.01	2.59	2	
19	2.370	2.06	2.65	5	
20	2.079	1.81	2.33	6	
21	2.282	1.98	2.56	4	
22	2.445	2.13	2.74	6	
23	2.157	1.88	2.42	5	accepted
24	2.505	2.18	2.80	8	rejected
25	2.404	2.09	2.69	7	accepted
26	2.152	1.87	2.41	6	
27	2.309	2.01	2.59	7	
28	2.198	1.91	2.46	7	
29	2.253	1.96	2.52	4	
30	2.412	2.10	2.70	7	
31	2.398	2.09	2.68	6	accepted

Table 16.7. Results for Stationarity Test A Applied to Case 2

Exper. Number	Results for Stationarity Test A Applied to $\pm 20\%$ Nonstationary Data - Case 3, Table 16.5				
	$\overline{x^2} \approx \sigma_x^2$	$50\%\ \chi^2$ interval		Number of Failures	Region of Acceptance
		lower limit $0.870\ \overline{x^2}$	upper limit $1.12\ \overline{x^2}$	q	$q < 8$
1	1.612	1.40	1.80	8	rejected
2	1.432	1.24	1.60	8	rejected
3	1.595	1.39	1.79	6	accepted
4	1.517	1.32	1.70	9	rejected
5	1.844	1.60	2.06	7	accepted
6	1.786	1.55	2.00	8	rejected
7	1.678	1.46	1.88	8	
8	1.769	1.54	1.98	8	
9	1.707	1.48	1.91	9	
10	1.616	1.40	1.81	8	rejected
11	1.804	1.57	2.02	6	accepted
12	1.762	1.53	1.97	7	accepted
13	1.731	1.50	1.94	8	rejected
14	1.729	1.50	1.94	10	
15	1.667	1.45	1.87	9	
16	1.852	1.61	2.07	9	rejected
17	1.582	1.38	1.77	6	accepted
18	1.730	1.50	1.94	8	rejected
19	1.621	1.41	1.82	6	accepted
20	1.731	1.50	1.94	7	accepted
21	1.826	1.59	2.04	4	accepted
22	1.853	1.61	2.08	9	rejected
23	1.706	1.48	1.91	8	rejected
24	1.712	1.49	1.92	7	accepted
25	1.717	1.49	1.92	7	accepted
26	1.754	1.52	1.96	8	rejected
27	1.724	1.50	1.93	8	
28	1.922	1.67	2.15	9	
29	1.708	1.48	1.91	9	
30	1.873	1.63	2.10	8	
31	1.836	1.60	2.06	9	rejected

Table 16.8. Results for Stationarity Test A Applied to Case 3

Exper. Number	Results for Stationary Test B Applied to Stationary Data - Case 1, Table 16.3					
	$\overline{x^2} \approx \sigma_x^2$	s'^2	$\hat{\epsilon}^2 = \frac{s'^2}{\left(\overline{x^2}\right)^2}$	$\epsilon^2 = \frac{1}{\sqrt{BT_s}}$	$\frac{\hat{\epsilon}^2}{\epsilon^2}$	Region of Acceptance $\frac{\hat{\epsilon}^2}{\epsilon^2} < 1.69$
1	2.296	0.171	0.0324	0.0333	0.97	accepted
2	2.428	0.106	0.0180		0.54	
3	2.413	0.285	0.0490		1.47	
4	2.324	0.249	0.0461		1.38	
5	2.491	0.117	0.0189		0.57	
6	2.143	0.181	0.0394		1.18	
7	2.285	0.099	0.019		0.57	
8	2.411	0.125	0.0215		0.64	
9	2.473	0.098	0.016		0.48	
10	2.474	0.137	0.0224		0.67	
11	2.498	0.240	0.0385		1.16	
12	2.419	0.117	0.0200		0.60	
13	2.256	0.100	0.0196		0.59	
14	2.320	0.184	0.0342		1.03	
15	2.334	0.198	0.0363		1.09	
16	2.336	0.199	0.0364		1.09	
17	2.170	0.148	0.0314		0.94	
18	2.458	0.130	0.0215		0.64	
19	2.449	0.140	0.0233		0.70	
20	2.388	0.110	0.0193		0.58	accepted
21	2.199	0.299	0.0618		1.86	rejected
22	2.493	0.167	0.0268		0.80	accepted
23	2.359	0.199	0.0358		1.08	
24	2.193	0.089	0.019		0.57	
25	2.456	0.077	0.013		0.39	
26	2.294	0.099	0.019		0.57	
27	2.414	0.151	0.0259		0.78	
28	2.583	0.190	0.0285		0.86	
29	2.390	0.081	0.014		0.42	
30	2.238	0.120	0.0240		0.72	
31	2.542	0.045	0.007	0.0333	0.21	accepted

Table 16.9. Results for Stationarity Test B Applied to Case 1

Exper. Number	Results for Stationary Test B Applied to $\pm 10\%$ Nonstationary Data - Case 2, Table 16.4					
	$\overline{x^2} \approx \sigma_x^2$	s^2	$\hat{\epsilon}^2 = \frac{s^2}{\left(\overline{x^2}\right)^2}$	$\epsilon^2 = \frac{1}{\sqrt{BT_s}}$	$\frac{\hat{\epsilon}^2}{\epsilon^2}$	Region of Acceptance $\frac{\hat{\epsilon}^2}{\epsilon^2} < 1.69$
1	2.277	0.277	0.0535	0.0333	1.61	accepted
2	2.311	0.310	0.0580		1.74	rejected
3	2.296	0.270	0.0512		1.54	accepted
4	2.512	0.272	0.0431		1.29	accepted
5	2.085	0.085	0.0195		0.58	accepted
6	2.475	0.470	0.0768		2.31	rejected
7	2.477	0.361	0.0588		1.76	rejected
8	2.385	0.177	0.0311		0.93	accepted
9	2.392	0.066	0.012		0.36	accepted
10	2.384	0.059	0.010		0.30	accepted
11	2.297	0.324	0.0614		1.84	rejected
12	2.365	0.081	0.014		0.42	accepted
13	2.103	0.118	0.0267		0.80	
14	2.282	0.208	0.0399		1.20	
15	2.239	0.123	0.0246		0.74	
16	2.120	0.065	0.014		0.42	
17	2.583	0.122	0.0183		0.55	
18	2.310	0.065	0.012		0.36	
19	2.370	0.164	0.0292		0.88	
20	2.079	0.170	0.0394		1.18	
21	2.282	0.077	0.015		0.45	accepted
22	2.445	0.502	0.0839		2.52	rejected
23	2.157	0.137	0.0295		0.88	accepted
24	2.505	0.204	0.0325		0.98	
25	2.404	0.223	0.0386		1.16	
26	2.152	0.151	0.0326		0.98	accepted
27	2.309	0.316	0.0593		1.78	rejected
28	2.198	0.192	0.0398		1.20	accepted
29	2.253	0.148	0.0291		0.87	accepted
30	2.412	0.369	0.0631		1.89	rejected
31	2.398	0.169	0.0294	0.0333	0.88	accepted

Table 16.10. Results for Stationarity Test B Applied to Case 2

Exper. Number	Results for Stationarity Test B Applied to $\pm 20\%$ Nonstationary Data - Case 3, Table 16.5					
	$\overline{x^2} \approx \sigma_x^2$	s^2	$\hat{\epsilon}^2 = \frac{s^2}{\left(\overline{x^2}\right)^2}$	$\epsilon^2 = \frac{1}{\sqrt{BT_s}}$	$\frac{\hat{\epsilon}^2}{\epsilon^2}$	Region of Acceptance $\frac{\hat{\epsilon}^2}{\epsilon^2} < 1.69$
1	1.612	0.281	0.108	0.0333	3.24	rejected
2	1.432	0.193	0.0941		2.82	
3	1.595	0.222	0.0873		2.62	
4	1.517	0.341	0.148		4.44	
5	1.844	0.662	0.195		5.86	
6	1.786	0.476	0.149		4.47	
7	1.678	0.392	0.139		4.17	
8	1.769	0.291	0.0930		2.79	
9	1.707	0.314	0.108		3.24	
10	1.616	0.185	0.0708		2.13	
11	1.804	0.197	0.0605		1.82	
12	1.762	0.259	0.0834		2.50	
13	1.731	0.290	0.0968		2.91	
14	1.729	0.230	0.0769		2.31	
15	1.667	0.386	0.139		4.17	
16	1.852	0.308	0.0898		2.70	
17	1.582	0.263	0.105		3.15	
18	1.730	0.175	0.0585		1.76	rejected
19	1.621	0.123	0.0468		1.40	accepted
20	1.731	0.242	0.0808		2.43	rejected
21	1.826	0.144	0.0432		1.30	accepted
22	1.853	0.464	0.135		4.05	rejected
23	1.706	0.230	0.0790		2.37	
24	1.712	0.286	0.0976		2.93	
25	1.717	0.295	0.100		3.00	
26	1.754	0.275	0.0894		2.68	
27	1.724	0.199	0.0670		2.01	
28	1.922	0.494	0.134		4.02	
29	1.708	0.179	0.0614		1.84	
30	1.873	0.284	0.0810		2.43	
31	1.836	0.391	0.116	0.0333	3.48	rejected

Table 16.11. Results for Stationarity Test B Applied to Case 3

Exper. Number	Results for Stationarity Test C Applied to Stationary Data - Case 1, Table 16.3			
	$\left(\overline{x^2}\right)$ max.	$\left(\overline{x^2}\right)$ min.	$\frac{\left(\overline{x^2}\right) \text{max.}}{\left(\overline{x^2}\right) \text{min.}}$	Region of Acceptance $\frac{\left(\overline{x^2}\right) \text{max.}}{\left(\overline{x^2}\right) \text{min.}} < 2.26$
1	3.20	1.78	1.80	accepted
2	3.05	2.00	1.52	
3	3.28	1.47	2.23	
4	3.43	1.54	2.23	
5	3.09	2.02	1.53	
6	2.74	1.40	1.96	
7	2.99	1.86	1.61	
8	3.18	1.89	1.68	
9	3.03	2.08	1.46	
10	3.25	1.79	1.82	
11	3.40	1.89	1.80	
12	2.89	1.62	1.78	
13	3.05	1.98	1.54	
14	3.35	1.64	2.04	
15	3.16	1.66	1.90	
16	2.98	1.33	2.24	
17	2.99	1.71	1.75	
18	3.09	1.98	1.56	
19	2.86	1.68	1.70	
20	2.92	1.74	1.68	accepted
21	3.63	1.37	2.65	rejected
22	3.10	1.63	1.90	accepted
23	3.00	1.72	1.74	
24	2.55	1.74	1.46	
25	2.90	1.90	1.53	
26	2.72	1.65	1.65	
27	3.10	1.67	1.86	
28	3.39	1.84	1.84	
29	2.76	1.75	1.58	
30	2.78	1.53	1.82	
31	2.81	2.13	1.32	accepted

Table 16.12. Results for Stationarity Test C Applied to Case 1

Exper. Number	Results for Stationarity Test C Applied to $\pm 10\%$ Nonstationary Data - Case 2, Table 16.4			
	$\left(\overline{x^2}\right)$ max.	$\left(\overline{x^2}\right)$ min.	$\frac{\left(\overline{x^2}\right) \text{max.}}{\left(\overline{x^2}\right) \text{min.}}$	Region of Acceptance $\frac{\left(\overline{x^2}\right) \text{max.}}{\left(\overline{x^2}\right) \text{min.}} < 2.26$
1	3.54	1.68	2.11	accepted
2	3.48	1.52	2.29	rejected
3	3.27	1.64	1.99	accepted
4	3.58	1.95	1.84	accepted
5	2.49	1.61	1.55	accepted
6	3.15	1.35	2.33	rejected
7	3.98	1.60	2.49	rejected
8	3.24	1.96	1.65	accepted
9	2.84	1.95	1.46	accepted
10	2.75	1.97	1.40	accepted
11	3.53	1.43	2.47	rejected
12	2.87	1.76	1.63	accepted
13	2.68	1.71	1.57	
14	3.10	1.54	2.01	
15	2.71	1.69	1.60	
16	2.50	1.73	1.44	
17	3.22	2.08	1.55	
18	2.82	1.89	1.49	
19	3.21	1.81	1.77	
20	2.92	1.66	1.76	
21	2.64	1.79	1.47	accepted
22	3.87	1.50	2.58	rejected
23	2.96	1.62	1.83	accepted
24	3.22	1.97	1.63	
25	3.32	1.86	1.78	
26	2.73	1.68	1.62	
27	3.15	1.59	1.98	
28	3.10	1.59	1.95	
29	3.03	1.61	1.88	
30	3.47	1.60	2.17	
31	2.97	1.80	1.65	accepted

Table 16.13. Results for Stationarity Test C Applied to Case 2

Exper. Number	Results for Stationarity Test C Applied to $\pm 20\%$ Nonstationary Data - Case 3, Table 16.5			
	$\left(\overline{x^2}\right)$ max.	$\left(\overline{x^2}\right)$ min.	$\dfrac{\left(\overline{x^2}\right)\text{ max.}}{\left(\overline{x^2}\right)\text{ min.}}$	Region of Acceptance $\dfrac{\left(\overline{x^2}\right)\text{ max.}}{\left(\overline{x^2}\right)\text{ min.}} < 2.26$
1	2.43	0.92	2.6	rejected
2	2.20	0.92	2.4	
3	2.33	0.89	2.6	
4	2.46	0.81	3.0	
5	3.92	0.93	4.2	
6	3.55	0.94	3.8	
7	2.96	0.99	3.0	
8	2.59	1.09	2.38	
9	2.48	1.07	2.32	rejected
10	2.30	1.06	2.17	accepted
11	2.62	1.25	2.10	accepted
12	2.54	0.97	2.6	rejected
13	2.50	0.93	2.7	
14	2.40	1.03	2.33	
15	2.59	0.81	3.2	
16	2.94	1.11	2.65	
17	2.38	0.83	2.9	rejected
18	2.29	1.17	1.96	accepted
19	2.06	1.00	2.06	accepted
20	2.53	1.08	2.34	rejected
21	2.68	1.24	2.16	accepted
22	2.97	0.98	3.0	rejected
23	2.61	1.09	2.39	
24	2.88	1.12	2.57	
25	2.82	1.04	2.71	rejected
26	2.70	1.20	2.25	accepted
27	2.40	1.04	2.31	rejected
28	3.02	1.10	2.74	rejected
29	2.33	1.22	1.91	accepted
30	2.63	0.96	2.7	rejected
31	3.00	1.03	2.91	rejected

Table 16.14. Results for Stationarity Test C Applied to Case 3

Summary of Stationarity Test Results:
Relative Power of Tests A, B, and C

$$\text{Empirical Value for } \beta = \frac{(\text{number of acceptances})}{(\text{number of experiments})}$$

$$\text{Power of Test} = 1 - \beta$$

Stationarity Test	± 10 Nonstationarity Case 2			± 20 Nonstationarity Case 3		
	Number of Acceptances	β	Power $(1 - \beta)$	Number of Acceptances	β	Power $(1 - \beta)$
A (Table 16.7, Table 16.8)	29	0.94	0.06	10	0.32	0.68
B (Table 16.10, Table 16.11)	24	0.77	0.33	2	0.06	0.94
C (Table 16.13, Table 16.14)	26	0.84	0.16	7	0.22	0.78

Table 16.15 Summary of Stationarity Test Results

16.5 DISCUSSION OF RESULTS

16.5.1 Relative Power of Stationarity Tests

It is seen from Table 16.15 that Test B is the most powerful and Test A is the least powerful of the three Stationarity Tests for the cases studied. These results are logical when one considers the amount of data employed by each test to reach a decision.

Test B effectively uses all the information that is available; namely, the actual value for each of the N number of mean square estimates obtained. Test C uses only two of the actual values, the maximum and minimum mean square measurements, plus the knowledge that the remaining (N - 2) mean square measurements fall between those two extreme values. Test A uses no actual values. The only information employed is the knowledge that a certain number q of the measurements fall outside a given χ^2 probability interval. Thus, the relative power of the three tests, as determined empirically in Table 16.15, is consistent with the amount of actual data employed by each test to reach a decision.

Now consider the three Stationarity Tests in terms of their relative desirability as measured by the combination of power and simplicity in application. Test A is the least powerful and also the most difficult to apply in actual practice. Thus, Test A is clearly the least desirable of the three. The relative desirability of Tests B and C is not so clear. Test B is the more powerful while Test C is simpler to apply. If the data analyst has a digital computer at his disposal to accomplish numerical computations with high speed, Test B would be the more desirable since the required computations would be no problem. However, if the data analysis is accomplished by hand, Test C might be considered more desirable because of its simplicity.

16.5.2 Application of Analysis of Variance Techniques

In Section 8, procedures are detailed for testing the equivalence of mean square measurements obtained from different flights. These techniques, which are the classical analysis of variance procedures, do not require a knowledge of $n = 2BT_1$. Furthermore, because the techniques employ all the data available from multiple flights, they detect with great power any common trend within flights as well as significant differences between flights.

The data presented in Section 16. 4 presents an excellent opportunity to illustrate the application of a one way analysis of variance procedure as a test for stationary in repeated flights.

Consider the mean square level data for nonstationarity Case 2, presented in Table 16. 4. Each experiment may be considered a flight where 10 mean square measurements were obtained during the flight. For the following example, experiments number 1, 2, and 3 are used to represent three similar flights where mean square measurements are obtained at similar times during the flights. The flights are tested for stationarity at the $\alpha = 0.05$ level of significance.

The appropriate data from Table 16. 4, is shown in Table 16. 16 in the desired format for a one way analysis of variance.

One Way Analysis of Variance 3 Flights, 10 Measurements per Flight			
Measurement Number	Flight 1	Flight 2	Flight 3
1	3. 54	2. 29	2. 51
2	1. 95	3. 48	2. 39
3	2. 86	2. 87	3. 27
4	1. 88	2. 45	2. 19
5	1. 97	2. 20	3. 04
6	2. 47	2. 49	2. 43
7	2. 00	1. 52	1. 70
8	2. 20	1. 70	1. 64
9	2. 22	2. 02	1. 82
10	1. 68	2. 09	1. 97

Table 16. 16 One Way Analysis of Variance Table

Using the computational procedures and notation outlined in Section 8, the following results are obtained.

$$R_1 = 8.34;\ R_2 = 7.82;\ R_3 = 9.00;\ R_4 = 6.52;\ R_5 = 7.21$$

$$R_6 = 7.39;\ R_7 = 5.22;\ R_8 = 5.54;\ R_9 = 6.06;\ R_{10} = 5.74$$

$$r = 10 \text{ rows};\ c = 3 \text{ columns};\ T = \sum_{i=1}^{10} R_i = 68.84$$

$$\frac{T^2}{30} = 158.0;\ \sum_{i=1}^{10} \sum_{j=1}^{3} x_{ij}^2 = 166.3;\ \sum_{i=1}^{10} R_i^2 = 3(162.8)$$

$$SS_3 = 162.8 - 158.0 = 4.8$$

$$SS = 166.3 - 158.0 = 8.3$$

$$SS_2 = 8.3 - 4.8 = 3.5$$

$$SS_3^* = \frac{SS_3}{(r - 1)} = 0.53 \text{ with } (r - 1) = 9 \text{ degrees of freedom}$$

$$SS_2^* = \frac{SS_2}{r(c - 1)} = 0.18 \text{ with } r(c - 1) = 20 \text{ degrees of freedom}$$

For $\alpha = 0.05$, $F_{(9, 20)} = 2.39$

$$\frac{SS_3^*}{SS_2^*} = 2.9 > 2.39$$

Thus, the hypothesis of equivalence for the measurements obtained during the flights is rejected at the 5% level of significance, and the flights are correctly interpreted as being nonstationary.

16.5.3 Combined Stationarity and Randomness Test

Referring to Randomness Test B developed in Section 15.1.3, it is seen that the principles and applications of Randomness Test B are quite similar to those for Stationarity Test B developed in this section. Both tests are based upon an analysis of the variability of mean square estimates. The only difference is that Randomness Test B employs a lower tail χ^2 test, as given by Eq. (15.26), while Stationarity Test B uses an upper tail χ^2 test, as given by Eq. (16.18). It is obvious that the two tests can readily be combined into a single hypothesis test for both randomness and stationarity.

Assume a collection of N independent mean square values, $\overline{x_i^2}$ $(i = 1, 2, 3, \ldots, N)$, are measured from a stationary random vibration response signal. Let the theoretical expected normalized variance ϵ^2, and the statistical estimated normalized variance $\hat{\epsilon}^2$ be computed using Eqs. (16.13) and (16.15), respectively. The following probability statement may be made.

$$\text{Prob}\left[\frac{\chi_{(1-\alpha/2)}}{N} \leq \frac{\hat{\epsilon}^2}{\epsilon^2} \leq \frac{\chi^2_{\alpha/2}}{N}\right] = (1 - \alpha) \qquad (16.39)$$

Thus, if it is hypothesized that the collection of mean square estimates were obtained from a random and stationary signal, the region of acceptance for a test of the double hypothesis at the α level of significance is given by

$$\frac{\chi_{(1-\alpha/2)}}{N} \leq \frac{\hat{\epsilon}^2}{\epsilon^2} < \frac{\chi^2_{\alpha/2}}{N} \qquad (16.40)$$

A plot of the acceptance regions of the normalized variance ratio for various values of α and N is shown in Figure 16.6.

It should be noted in Eq. (16.40) that the upper and lower χ^2 limits are based upon $\alpha/2$ rather than α as in Randomness Test B or Stationarity Test B. Thus, the upper limit of Eq. (16.40) is higher than the limit for Stationarity Test B alone, as given by Eq. (16.18).

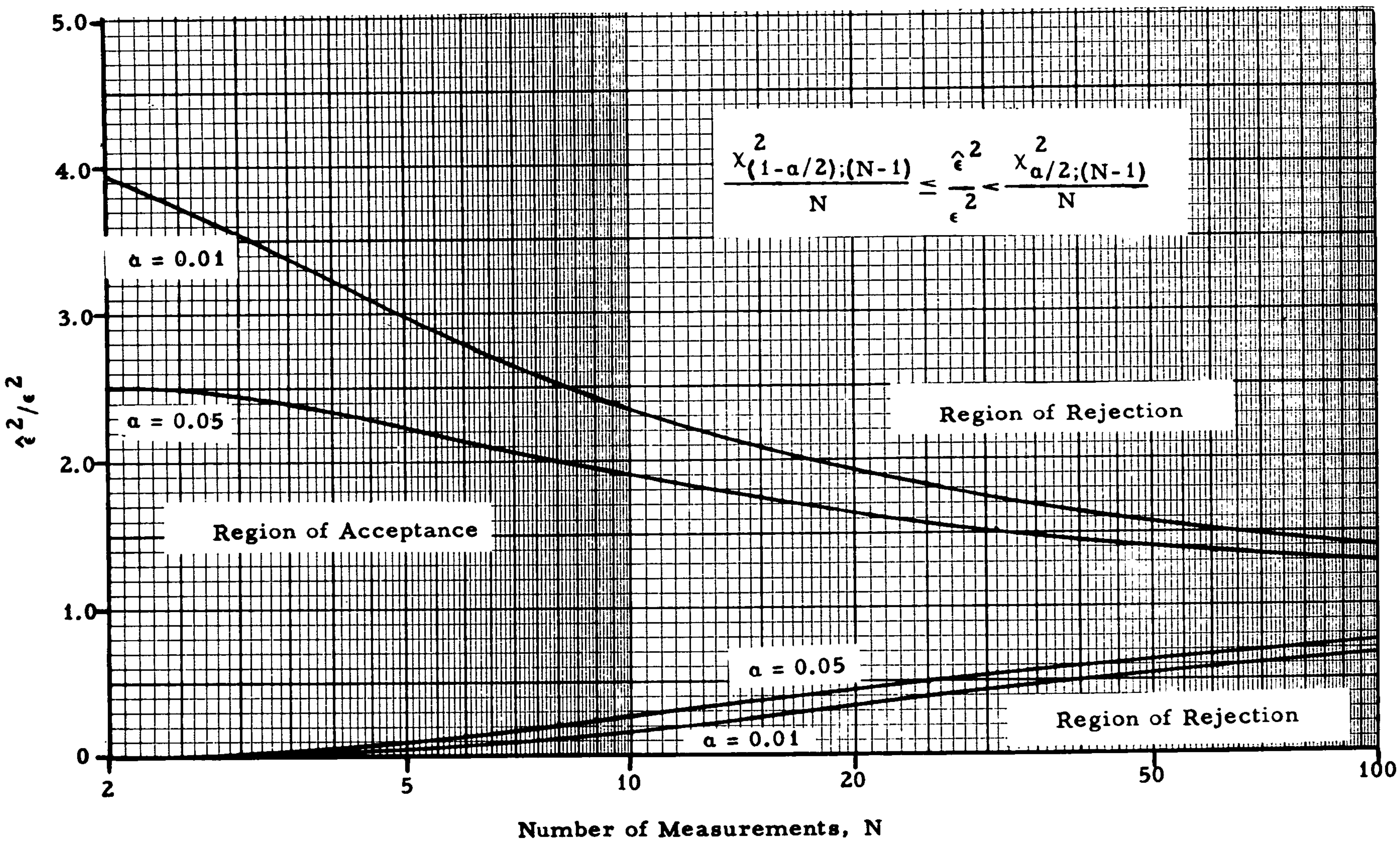

Figure 16.6 Acceptance Regions for Combined Randomness and Stationarity Test

Similarly, the lower limit of Eq. (16. 40) is lower than the limit for Randomness Test B alone, as given by Eq. (15. 26). Hence, application of the combined test at a given level of significance α will involve a wider region of acceptance than for the two individual tests. This is to be expected since the probabilities of Type I errors for the individual tests are effectively pooled into a single Type I error probability for the combined test. The net result is that more measurements are required for the combined test than for each individual test if the same Type II error limitations are to be maintained.

The combined test for randomness and stationarity does pose one practical consideration. Randomness Test B is most effective when applied to narrow frequency ranges obtained by locating a narrow bandwidth filter over a sharp peak in the power spectrum of the signal. On the other hand, Stationarity Test B, as any test for stationarity, should ideally be applied to the entire frequency range of the signal. If no peaks appear in the power spectrum, a quantitative test for randomness is usually not needed as is discussed in Section 15. 5. 3. For this case, the combined test is not really applicable. However, if one or more sharp peaks are present in the power spectrum, the signal should be tested for both randomness and stationarity. For this case, it is often acceptable to apply the combined test to the narrow frequency range of the peaks, as would be done with Randomness Test B alone. Most of the relative power of the vibration response represented by the signal being analyzed will be in the sharp power spectral density peaks. If the peaks in the power spectrum are shown to be stationary, it is reasonable to assume the entire signal is stationary for most engineering applications.

16.6 CONCLUSIONS

Three different procedures for testing single sample records for stationarity have been experimentally evaluated. One of the procedures (Test A) is the stationarity test originally proposed in Section 6.1.8 of Ref. [1]. The other two procedures (Tests B and C) are based upon more recent concepts developed herein.

The experimental results confirm the validity of the theory for Stationarity Test A, as developed in Ref. [1]. However, the results also indicate Test A is less powerful and more complicated than the other two tests proposed in this section. Thus, it is recommended that Stationarity Test A from Ref. [1] be dropped in favor of either of the two additional tests for stationarity presented here.

For situations of repeated flights (multiple points) where a number of different sample records are available, a fourth test for stationarity is illustrated herein based on one way analysis of variance procedures, presented in Section 8. The results demonstrate that analysis of variance procedures will detect nonstationarities in repeated experiments with great power.

16.7 REFERENCES

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2. Dixson, W. J., and F. J. Massey, Jr., Introduction to Statistical Analysis, Second Edition, McGraw-Hill Book Company, Inc. New York. 1957.

17. TESTS FOR NORMALITY

17.1 THEORY OF TESTS FOR NORMALITY

17.1.1 General Remarks

A random signal is said to be normal or Gaussian when it has a specific amplitude probability density function as defined by Eq. (11.4) in Section 11.4.2. For signals with a mean value $\mu_x = 0$, Eq. (11.4) becomes

$$p_0(x) = \frac{1}{\sigma_x\sqrt{2\pi}} \; e^{-x^2/2\sigma_x^2} \qquad (17.1)$$

where σ_x^2 is the mean square value and σ_x is the root mean square (rms) value for the signal.

Two quantitative procedures for testing a single random signal record for normality are suggested in Ref. [1]. The first procedure, which is outlined in Section 5.3.2 of Ref. [1], is the classical "chi-squared goodness of fit" test for normality. The "goodness of fit" test is hereafter referred to as Normality Test A.

The second procedure, which is proposed in Section 6.1.10 of Ref. [1], involves the analysis of analog probability density estimates. More specifically, a probability interval for the values of a measured probability density estimate $\hat{p}(x)$ is established based upon the assumption that the true probability density function being measured is Gaussian. The number of times that the estimate $\hat{p}(x)$ falls outside this probability interval (a failure) for measurements at many different amplitudes x is counted. By considering the number of failures to have a binomial sampling distribution, a hypothesis of normality may be tested. The general procedure is identical in concept and application to the procedure developed and studied in Section 16 as Stationarity Test A. The validity of the general procedure is substantiated in Section 16. However, Stationarity Test A in Section 16 was found to be relatively complicated to apply and weak in terms of detecting conditions of nonstationarity. Because of the strong similarity in concept and application, the same conclusions would undoubtedly apply to the test for normality proposed in Section 6.1.10 of Ref. [1].

A practical modification of the normality test originally suggested in Section 6.1.10 is proposed here. This test involves the individual testing

of probability density estimates at different amplitudes, and is hereafter referred to as Normality Test B.

17.1.2 Review of Normality Test A

The simplest test for normality is the classical chi-squared "goodness of fit" test. The "goodness of fit" test is applicable to amplitude values from the sample record and, thus, requires no special measurements or instruments. It is necessary only to reduce the amplitude time history from the sample record to a digital form. It is not necessary to discuss the theory and application of the "goodness of fit" test here since these discussions are available with a detailed illustration in Section 5.3.2 of Ref. [1], as well as in most statistics books. Furthermore, the application of the "goodness of fit" test to real data is illustrated in Section 14.4.5 of this report, where it is employed to test a collection of probability density estimates for normality.

17.1.3 Principles of Normality Test B

Consider a stationary random signal $x(t)$ which hypothetically exists over all time. Let $x(t)$ have an amplitude probability density function $p(x)$ with a mean value of zero. Further assume that the equivalent ideal frequency bandwidth B is at least one-third the number of zero crossings per second $\overline{\nu}_0$. Then, a measured probability density estimate $\hat{p}(x)$ will have a normalized variance and standard error as given in Section 14.5 as follows.

$$\epsilon^2 = \frac{0.068}{\Delta x\, \hat{p}(x)\, \overline{\nu}_0\, T_1} \tag{17.2a}$$

$$\epsilon = \frac{0.26}{\sqrt{\Delta x\, \hat{p}(x)\, \overline{\nu}_0\, T_1}} \tag{17.2b}$$

As in Section 14, Δx is the amplitude window used for the analysis, $\hat{p}(x)$ is the probability density measured at an amplitude x, $\overline{\nu}_0$ is the number of zero crossings per second, and T_1 is the averaging time.

It is shown in Section 14 that probability density estimates may be considered to be normally distributed about the true probability density at any specified amplitude. Then, a measurement $\hat{p}_v$ for the specific

amplitude $x = v$ will have a sampling distribution given by

$$(\hat{p}_v - p_v) \sim \text{Normal}(\epsilon\, p_v) \qquad (17.3)$$

where "$\sim$" means "distributed as," and Normal $(\epsilon\, p_v)$ is a normal distribution with a mean of zero and a standard deviation of $\epsilon\, p_v$ (a variance of $\epsilon^2 p_v^2$). From Eq. (17.3), the following probability statement can be made.

$$\text{Prob}\left[-(\epsilon\, p_v)\, z_{\alpha/2} \leq (\hat{p}_v - p_v) \leq (\epsilon\, p_v)\, z_{\alpha/2}\right] = (1 - \alpha) \qquad (17.4)$$

where $z_{\alpha/2}$ is the normal deviate for $\alpha/2$.

Let it be hypothesized that a sample random signal record has a Gaussian amplitude probability density function, $p_o(x)$. If this is true, $p(x) = p_o(x)$, and the hypothesis H_0 is

$$H_0 \;:\; \hat{p}_v = p_{ov} \quad \text{for any amplitude } x = v \qquad (17.5)$$

If the signal is not Gaussian, $\hat{p}_v$ may be greater or less than p_{ov} as defined in Eq. (17.1), which is why a two-sided probability statement is used in Eq. (17.4). Then H_0 is tested by computing ϵ in Eq. (17.2b) using p_{ov}, and comparing the difference $(\hat{p}_v - p_{ov})$ to the limits in Eq. (17.4) for any desired level of significance α. The region of acceptance for H_0 is

$$\left|\hat{p}_v - p_{ov}\right| \leq (\epsilon\, p_{ov})\, z_{\alpha/2} \qquad (17.6)$$

If the absolute value of the difference, $|\hat{p}_v - p_{ov}|$, is less than the noted limit, H_0 is accepted and the random signal is considered to have a Gaussian probability density at the amplitude $x = v$. If $|\hat{p}_v - p_{ov}|$ is greater than the noted limit, H_0 is rejected and there is reason to suspect that the signal is not Gaussian at the amplitude $x = v$. The hypothesis H_0 may be tested for as many different amplitudes x as desired.

The probability of a Type I error is α for each amplitude tested. The probability of a Type II error, β, is a function of the level of

significance α, the standard deviation $\epsilon\, p_{ov}$, and the non-Gaussian probability density p_v which one wishes to detect with a probability of β. Unfortunately, conventional Operating Characteristics (OC) curves are not readily applicable to this problem because the measurement standard deviation is given by a composite value $\epsilon\, p_{ov}$ rather than by a distinct population standard deviation and sample size. However, for a normal sampling distribution, the risk of a Type II error for detecting a difference $|p_v - p_{ov}| = (\epsilon\, p_{ov})\, z_{\alpha/2}$ is always $\beta = 0.50$. This point is illustrated in Figure 17.1. It is clear that $\beta = 0.50$ since exactly half of the normal distribution is covered at either extreme.

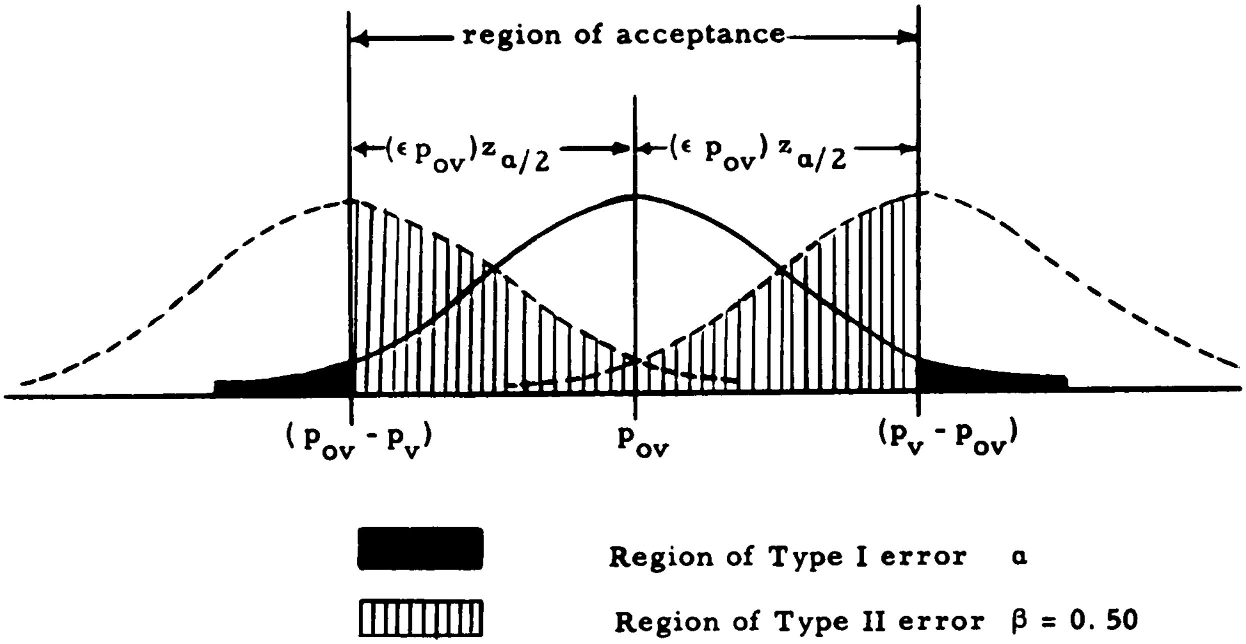

Figure 17.1 Probable Type I and Type II Errors for Normality Test B

It is important to note that the α and β risks are determined by $(\epsilon\, p_{ov})$ which in turn is determined by the record length T and the amplitude $x = v$ at which the test is to be applied. Thus, if a predetermined α and β risk are desired, these matters must be considered to arrive at the necessary record length T to be obtained.

One problem remains to be resolved. In practice, Normality Test B will actually require that many different amplitudes be tested as part of one general test for normality of a single sample record. To permit an easy test decision, it is desirable that a general hypothesis of normality be rejected if the test at any one individual amplitude produces a rejection. However, if N number of amplitudes are tested, each at the α level of significance, it is obvious that the probability of at least one Type I error is considerably greater than α. This probability of at least one Type I error, denoted by α', is developed as follows.

Assume a signal with a true Gaussian probability density function is sampled and tested for normality at N number of different amplitudes. If each amplitude is tested at the α level of significance, the probability of correctly accepting a normality hypothesis at each amplitude is, of course, $(1 - \alpha)$. Thus, the probability of accepting the normality hypothesis at all N amplitudes is $(1 - \alpha)^N$. It follows that the probability of making at least one Type I error for the N number of tests is

$$\alpha' = 1 - (1 - \alpha)^N \tag{17.7}$$

Equation (17.7) defines the over-all probability of a Type I error for Randomness Test B. If a Type I error probability of α' is desired for the over-all test, each of the N number of amplitudes must be tested at the α level of significance given by

$$\alpha = 1 - (1 - \alpha')^{1/N} \tag{17.8}$$

In summary, the procedure for applying Normality Test B is as follows.

1. Establish the desired value of α', the risk of making a Type I error, and the number of amplitudes N which are to be tested. Compute the level of significance α for the test at each amplitude using Eq. (17.8).
2. Establish the deviation from normality, $|p_v - p_{ov}|$, which is to be detected at each amplitude with a probability of $\beta = 0.50$.

3. Using a table of normal deviates, determine the value for $z_{\alpha/2}$.
4. Using a table of normal ordinates, determine the Gaussian probability p_{ov} for the maximum amplitude $x = v(\max)$ to be tested.
5. Determine the required value for ϵ by letting $|p_v - p_{ov}| = (\epsilon\, p_{ov})\, z_{\alpha/2}$.
6. Solve for T_1 in Eq. (17.2) using $p(x) = p_{ov}$ and an estimate for $\bar{\nu}_0$. The sample record to be obtained must have a length of $T \geq T_1$.
7. Measure the probability density estimates $\hat{p}_v$ from the sample record at the N number of required amplitudes, and test the hypothesis H_0 in Eq. (17.5) using the acceptance region in Eq. (17.6).
8. If a rejection is obtained at any one of the N number of amplitudes tested, there is reason to believe the sample record was not obtained from a signal with a Gaussian probability density function.

Example:

The above discussions will be clarified by considering a specific example. The example used is an actual case which is studied herein experimentally.

Assume a signal representing a stationary random vibration response is to be sampled so that a test for normality may be performed for amplitudes out to 2.5 times the root mean square value for the signal ($2.5\ \sigma_x$). The expected number of zero crossings per second for the signal is $\bar{\nu}_0 = 788$. The amplitude window width of the analyzer to be used is 0.1 for an input signal level of σ_x = unity. Symmetry is assumed so that only $N = 6$ positive amplitudes will be tested ($v = 0, 0.5\sigma_x, 1.0\sigma_x, 1.5\sigma_x, 2.0\sigma_x$, and $2.5\sigma_x$). It is desired that the Type I error probability for the entire test be $\alpha' = 0.05$, and that a ± 25% deviation from normality at any of the six amplitudes be detected with $\beta = 0.50$ probability of a Type II error.

From Eq. (17.8), for $\alpha' = 0.05$ and $N = 6$, the required level of significance for the test at each amplitude is $\alpha \approx 0.01$. From a table of normal deviates (Ref. [2], Table II), $z_{\alpha/2} = 2.6$ for $\alpha \approx 0.01$. From a table of normal ordinates (Ref. [2], Table I), for $v = 2.5\sigma_x$, $p_{ov} = 0.0175$. Thus, for a ± 25% deviation, $|p_v - p_{ov}| = 0.25\, p_{ov}$, and the required value of ϵ is given by

$$\epsilon = \frac{|p_v - p_{ov}|}{p_{ov}\, z_{\alpha/2}} = \frac{0.25\, p_{ov}}{2.6\, p_{ov}} = 0.1$$

From Eq. (17.2), the averaging time T_1 required for the maximum amplitude ($v = 2.5\sigma_x$) is determined to be

$$T_1 = \frac{0.068}{\epsilon^2\, \Delta x\, p_{ov}\, \bar{\nu}_0} = \frac{0.068}{(0.01)(0.1)(0.0175)(788)} = 5 \text{ seconds}$$

Thus, a sample record length of at least T = 5 seconds is required. It is interesting to note that if the test were to be performed with the same α' and β risks out to $3\sigma_x$, the required record length would be $T \approx 20$ seconds. Out to $4\sigma_x$, the required record length would be $T \approx 45$ seconds.

To actually perform the test for normality, a probability density estimate $\hat{p}_v$ must be measured at each of the six amplitudes to be tested. The region of acceptance for the normality hypothesis H_0 at each amplitude v is

$$|\hat{p}_v - p_{ov}| \leq (\epsilon\, p_{ov})\, z_{\alpha/2} = 0.26\, p_{ov}$$

Note that the measured estimates $\hat{p}_v$ should all have the same normalized variance ($\epsilon^2 = 0.01$) if the same Type I and Type II error limits are to be maintained at all amplitudes. However, ϵ^2 is inversely proportional to $\hat{p}_v$. Thus, from Eq. (17.2), the required averaging time T_1 becomes shorter as the amplitude being tested becomes smaller. For the example at hand, the required averaging times and acceptance regions are summarized in Table 17.1.

Limits of Acceptance Region for $H_0: \hat{p}_v = p_{ov}$ $\Delta x = 0.1\sigma_x$; $\overline{\nu}_0$ = 788 crossings per second; $\alpha' = 0.05$; $\alpha \approx 0.01$ $\beta = 0.50$ for detecting a 25% deviation from normality				
Amplitude v	Gaussian Probability p_{ov}	Normalized Standard Error ϵ	Required Averaging Time T_1 seconds	Limit for Acceptance Region $0.26\, p_{ov}$
0	0.399	0.1	0.22	0.10
$0.5\sigma_x$	0.352	0.1	0.24	0.092
$1.0\sigma_x$	0.242	0.1	0.36	0.063
$1.5\sigma_x$	0.130	0.1	0.66	0.034
$2.0\sigma_x$	0.0540	0.1	1.6	0.014
$2.5\sigma_x$	0.0175	0.1	5.0	0.0045

Table 17.1 Limits of Acceptance Region for Normality Test B

17.2 DESIGN OF EXPERIMENTS AND PROCEDURES

A comprehensive experimental study of the two normality tests discussed in Section 17.1 is not really feasible since a random signal which has an exact Gaussian probability density function at all amplitudes is unattainable. However, there is little reason to question either the theoretical foundation or practical applicability of the two tests. Normality Test A (chi-squared goodness of fit test) is a classical procedure which has been widely applied for years to many problems including digital vibration data analysis. Normality Test B is based solely upon the uncertainty of analog probability density estimates which is studied in detail in Section 14.

It will be sufficient here to illustrate Normality Test B by applying the test to probability density data gathered in Section 14. Specifically, consider the data presented under A-3 in Table 14.2 and B-1 through B-5 in Table 14.3. These data represent probability density estimates for amplitudes of $v = 0$ (A-3), $0.5\sigma_x$ (B-1), $1.0\sigma_x$ (B-2), $1.5\sigma_x$ (B-3), $2.0\sigma_x$ (B-4), and $2.5\sigma_x$ (B-5) with the following test parameters.

1. $\sigma_x = 1.00$ volts
2. $\Delta x = 0.100$ volts
3. $f_a = 95$ cps ; $f_b = 630$ cps

$$\overline{\nu}_0 = 2\sqrt{\frac{f_a^2 + f_a f_b + f_b^2}{3}} = 788 \text{ crossings per second}$$

4. $T_1 = 2K = 0.218$ seconds per estimate

Let it be hypothesized that the random noise generator used for the experiments in Section 14 (Item A, Table 14.1) produces a random signal with a true Gaussian probability density function. The hypothesis

of normality, H_0, is to be tested at six amplitudes from 0 to 2.5 volts (σ_x = unity) at the $\alpha' = 0.05$ level of significance. It is desired that a 25% deviation from a Gaussian probability density be detected with a probability of $\beta = 0.50$. Note that these requirements and parameters are exactly the same as used for the example in Section 17.1. From that example, the required averaging times and the resulting acceptance regions for H_0 are as given in Table 17.1.

An estimate $\hat{p}_v$ averaged over the desired time intervals may be obtained from Table 14.2 and 14.3 by averaging together as many estimates based on $T_1 = 0.218$ seconds as required for each of the six amplitudes of interest.

17.3 INSTRUMENTATION

The instruments and test set-up are detailed in Section 14.3.

17.4 RESULTS OF EXPERIMENTS

17.4.1 Probability Density Data

The probability density estimates for the six amplitudes of interest are presented in Table 17.2. Each estimate $\hat{p}_v$ is obtained by averaging, from the top down, the required number of values presented in the appropriate columns of Tables 14.2 and 14.3 of Section 14.

Probability Density Estimates p_v for Different Amplitudes v
$\sigma_x = 1.00$ volts; $\Delta x = 0.100$ volts
$\bar{\nu}_0 = 788$ crossings per second

Amplitude v (volts)		probability density p_v
0	(A-3, Table 14.2)	0.385
0.5	(B-1, Table 14.3)	0.400
1.0	(B-2, Table 14.3)	0.260
1.5	(B-3, Table 14.3)	0.101
2.0	(B-4, Table 14.3)	0.0418
2.5	(B-5, Table 14.3)	0.0141

Table 17.2 Probability Density Data for Normality Test B

17.4.2 Test for Normality

Using the data presented in Tables 17.1 and 17.2, the random noise generator is tested for normality by application of Normality Test B at the $\alpha' = 0.05$ level of significance, as summarized in Table 17.3.

Results of Test for Normality of Signal from Random Noise Generator					
amplitude v	Estimated Probability Density Table 17.2 $\hat{p}_v$	Gaussian Probability Density Table 17.1 p_{ov}	$\lvert \hat{p} - p_{ov} \rvert$	Limit of Acceptance Region for $\lvert \hat{p} - p_{ov} \rvert$ Table 17.1	Test of $H_0: \hat{p}_v = p_{ov}$
0	0.385	0.399	0.014	0.10	accepted
0.5	0.400	0.352	0.048	0.092	accepted
1.0	0.260	0.242	0.018	0.063	accepted
1.5	0.101	0.130	0.029	0.034	accepted
2.0	0.0418	0.0540	0.0122	0.014	accepted
2.5	0.0141	0.0175	0.0034	0.0046	accepted

Table 17.3 Results of Test for Normality

From Table 17.3, it is seen that the hypothesis of normality at the six specific amplitudes tested is accepted. Thus, the general hypothesis of normality for the random signal is accepted.

17.5 DISCUSSION OF RESULTS

The results presented in Table 17.3 indicate the random noise generator employed for the experiments in this report produces a signal which is Gaussian within the Type II error limits of the normality test performed ($\beta = 0.50$ for detecting a 25% deviation from normality at each amplitude tested). This does not necessarily mean that the signal from the noise generator is exactly Gaussian. It is quite possible that a hypothesis of normality would be rejected if the test were performed with a more stringent Type II error limitation. However, for the experiments performed in this report, the Type II error requirements used for the normality test in Section 17.4.2 are sufficient to justify the qualitative statement that the noise generator produces an "approximately Gaussian" signal.

A plot of the probability density function for the signal from the random noise generator is presented in Figure 14.3 of Section 14. The ideal Gaussian probability density function is shown on the same scale. The results there clearly show the strong Gaussian tendencies of the noise generator signal.

17.6 CONCLUSIONS

Two different procedures for testing single sample records for normality have been reviewed. One of the procedures (Test A) is the "chi-squared goodness of fit" test outlined in Section 5.3.2 of Ref. [1]. The other procedure (Test B) is a modification of the normality test originally proposed in Section 6.1.10 of Ref. [1].

Normality Test A is applicable to amplitude time history data in digital form. Normality Test B is applicable to analog probability density estimates. The probability density data obtained for experiments in Section 14 is used here to illustrate the application of Test B to real data. Both procedures are believed to be practical for application to flight vehicle vibration data.

17.7 REFERENCES

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18. CONCLUSIONS AND RECOMMENDATIONS FROM EXPERIMENTAL PROGRAM

The general objectives of the studies presented in this report and a brief summary of main results are stated in Section 1. Part I of the report, consisting of the first ten sections, is concerned with analytical studies which are reviewed in the conclusions and recommendations of Section 10. Part II of the report, consisting of Sections 11 through 18, is concerned with experimental studies which are terminated by the conclusions and recommendations presented here. An introduction to the experimental program, including the necessary background material, is presented in Section 11. These experimental studies are based upon data analyzed by analog instruments. However, the results and conclusions are directly applicable to data analyzed using digital techniques, except where noted. It should also be recognized that the statistical procedures employed to evaluate the experimental results have broad applications to many general data analysis problems for other physical areas.

18.1 REVIEW OF EXPERIMENTAL PROGRAM

Section 12. Uncertainty of Zero Crossing Estimates

Theoretical studies are reviewed which deal with the application of statistical run theory to the prediction of the sampling distribution for random signal zero crossings. Experiments are performed involving three types of signals; (1) random signals with uniform power spectra, (2) random signals with non-uniform power spectra, (3) nonrandom signals consisting of a sine wave plus noise. Experimental zero crossing measurements are compared to theoretical predictions by carefully designed statistical tests.

The results confirm that run theory is applicable to the zero crossing problem. The sampling mean and variance for the zero crossings of a random signal are predicted correctly when the signal has a reasonably uniform power spectrum. Furthermore, the presence of a sine wave in an otherwise random signal significantly alters the resulting zero crossings, which indicates possible applications of run theory to a test for randomness. However, sharp peaks in the power spectrum of a random signal also alters the resulting zero crossings.

An important conclusion of this section is as follows. Given a stationary random signal with a uniform power spectrum between the frequencies f_a and f_b, the sampling distribution for the zero crossings

ν_0 measured from a sample record of length T (seconds) will have a mean and variance

$$\mu_{\nu_0} = \frac{n}{2}$$

$$\sigma^2_{\nu_0} = \frac{\left(\frac{n}{2} - 1\right)n}{2(n-1)}$$

where

$$n = 2\bar{\nu}_0 T$$

$$\bar{\nu}_0 = 2\sqrt{\frac{f_a^2 + f_a f_b + f_b^2}{3}}$$

Section 13. Uncertainty of Power Spectra (Mean Square) Estimates

Theoretical studies are reviewed which deal with the sampling distribution for power spectra estimates measured from sample records of random signals. Experiments are performed involving analog power spectra measurements for a wide range of frequency bandwidths, center frequencies, and averaging times. The experimental results are in complete agreement with the theoretical prediction for the variance of power spectra estimates.

An important conclusion of this section is as follows. Given a stationary random signal, the sampling distribution for a power spectral density estimate $\hat{G}(f)$ measured from a sample record of length T (seconds) will have a normalized variance

$$\epsilon^2 = \frac{\sigma^2_{\hat{G}(f)}}{G^2(f)} \approx \frac{1}{BT}$$

Here, B is the frequency bandwidth in cps of the narrow band pass filter used for the analysis.

The study of power spectra measurements is extended to cover mean square level estimates. General expressions are developed for the sampling distribution variance of wide bandwidth mean square value estimates.

Section 14. Uncertainty of Probability Density Estimates

Previous theoretical studies are reviewed and extended for the sampling distribution of amplitude probability density estimates. Two slightly different theoretical expressions for the variance of probability density estimates are obtained. Experiments are performed involving analog probability density measurements for a wide range of frequency characteristics and averaging times.

For many practical conditions, the experimental results indicate that an appropriate expression for the normalized variance of probability density estimates $\hat{p}(x)$ measured from a sample record of length T (seconds) is as follows.

$$\epsilon^2 = \frac{\sigma^2_{\hat{p}(x)}}{p^2(x)} \approx \frac{0.07}{\Delta x \, \hat{p}(x) \, \overline{\nu}_0 \, T}$$

Here, Δx is the width of the amplitude window used for the analysis, and $\overline{\nu}_0$ is the expected number of zero crossings per second.

For those cases where the random signal has a narrow bandwidth, the experimental results indicate that a more valid representation for the normalized variance of probability density estimates is as follows.

$$\epsilon^2 = \frac{\sigma^2_{\hat{p}(x)}}{p^2(x)} \approx \frac{0.03}{\Delta x \, \hat{p}(x) \, BT}$$

Here, B is the equivalent ideal frequency bandwidth in cps.

It is believed that the true expression for the sampling distribution variance of probability density estimates is more general, and perhaps involves more terms than the expressions developed and studied in this section. These matters are not completely resolved.

Section 15. Tests for Randomness

Two different quantitative procedures for testing individual sample records for randomness are studied. One of the procedures (Test A) is the randomness test proposed originally, and is based upon an analysis of the zero crossings measured from a sample record. The second procedure (Test B) is based upon theoretical ideas developed herein involving an analysis of the variability of mean square value measurements. Experimental data gathered for the studies in Section 12, as well as additional experiments performed in this section, are employed to confirm the basic theory of the two tests and to evaluate their relative effectiveness.

The experimental results are in agreement with theoretical predictions. Randomness Test A is shown to be a powerful technique for detecting sine waves in an otherwise random signal, under ideal conditions. However, the results also indicate Test A will erroneously reject random signals as being nonrandom unless the power spectra of the signals are reasonably uniform. This fact makes the application of Test A to flight vehicle vibration impractical in many cases, since such data rarely have uniform power spectra. Randomness Test B proposed in this section does not present the above problem. Although Test B is less powerful for detecting sine waves, its application to flight vehicle vibration data is more practical. It is recommended that Randomness Test A be dropped in favor of the new Test B presented herein.

Qualitative techniques for evaluating the randomness of flight vehicle vibration data are also discussed. These techniques involve observation of certain distinguishing details in power spectral density functions, autocorrelation functions, and amplitude probability density functions. These qualitative procedures will often reveal periodicities in an otherwise random vibration without the need for quantitative testing of the data.

Section 16. Tests for Stationarity

Three separate quantitative procedures for testing individual sample records for stationarity are studied. One of the procedures (Test A) is the stationarity test proposed originally. The other two procedures (Tests B and C) are based upon concepts developed herein. All three procedures involve an analysis of the variability of mean square value measurements. Extensive experiments are performed to confirm basic theoretical ideas and to evaluate the relative effectiveness of the three tests.

The experimental results are in agreement with theoretical predictions. The results also indicate that Stationarity Test A is the least powerful and the most complicated in application of the three tests. It is recommended that Stationarity Test A be dropped in favor of either of the two additional tests presented herein.

For situations where data is collected from repeated flights (or from multiple points) such that a number of different sample records are available, a fourth test for stationarity is illustrated. This test is based upon one-way analysis of variance procedures, as presented in Section 8 of this report. The results demonstrate that these analysis-of-variance procedures will detect nonstationary conditions in repeated experiments with great power.

Section 17. Tests for Normality

Two different quantitative procedures for testing individual sample records for normality are considered. One procedure (Test A) is the "chi-squared goodness of fit" test. The other procedure (Test B) is a practical modification of the test for normality proposed originally. Test A is applicable to digital amplitude time history data, while Test B is applicable to analog probability density data.

Normality Test A (goodness-of-fit test) is a classical procedure which requires no verification. The theoretical basis for Test B is substantiated by experiments performed in Section 14. Experimental probability density data gathered in Section 14 is employed to illustrate the practical application of Normality Test B.

18.2 RECOMMENDATIONS FOR FURTHER EXPERIMENTAL AND THEORETICAL WORK FROM PART II

1. The analysis of extreme and peak amplitude values is an area in need of further study. Considerable theoretical and experimental work is required to develop suitable techniques for estimating extreme and peak values from relatively short sample records. These matters are of major importance to the general problem of flight vehicle structural reliability, including the prediction of catastrophic structural failures due to extreme loads as well as long term structural damage due to fatigue.

2. The theoretical formulas for the sampling distribution of amplitude probability density estimates, as developed in Section 14 of this report, appear to be incomplete. Further fundamental study is needed, along with appropriate experimental verification. The discussions and data given in Section 14 present a sound foundation for such future work.

3. Procedures for detecting a periodic component with fixed frequency in an otherwise random signal are well developed in Section 15 of this report. However, the related problem of detecting a periodic component with changing frequency is not considered. More theoretical study of this special situation is needed, followed by appropriate experimental verification.

4. A comprehensive experimental and theoretical program should be conducted on the accuracy of transfer function measurements. This will require development of tests for linearity, based upon theoretical uncertainty formulas for measuring coherence functions. Transfer function measurements are important in making predictions of input-output relations for linear systems. Inaccurate estimates of transfer functions can lead to serious design problems.

5. The sampling distribution for correlation measurements is well defined for time displacements near zero, since the correlation function at zero delay is simply a mean square value. However, the sampling distribution for correlation measurements at large time displacements is not well defined. Additional theoretical studies of this problem are needed, followed by experimental verification.

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This list of references does not include the detailed list compiled at the end of Section 9, which is a summary of the response of nonlinear systems to random excitation.

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This report presents results from theoretical and experimental studies on random signal estimation and measurement pertinent to flight vehicle vibration problems. The report is divided into two self-contained parts. Part I, Theoretical Studies of Random Signal Estimation, develops new basic mathematical ideas for nonstationary data analysis. Discussion includes methods for estimating mean and mean square values of nonstationary data, correlation and spectral properties of nonstationary

(over)

1. Vibration
2. Statistics
3. Vibration Measurements

I. AFSC Project 1370, Task 137009
II. AF33(657)-7459
III. Thompson Ramo Wooldridge Inc., RW Division Canoga Park, Calif.
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data, and various input-output relations for passage of nonstationary data through linear systems. Also discussed are sampling formulas for gathering an optimum amount of data, analysis of variance procedures applied to vibration data, and a summary of nonlinear systems response to random excitation.

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